Fractional Quantum Hall effect in a Curved Space

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See M. Laskin's poster for more information

Outline

- Introduction to Quantum Hall effect

- Physics of the Lowest Landau Level in curved space
  - ➡ Linear response relations

- Ward Identity (i.e. loop equation) for fractional quantum Hall states
  - ➡ Particle density and generating functional

Goal and Motivation

Compute ground state correlation functions for arbitrary metric and inhomogeneous fields (hence FQH in curved space)

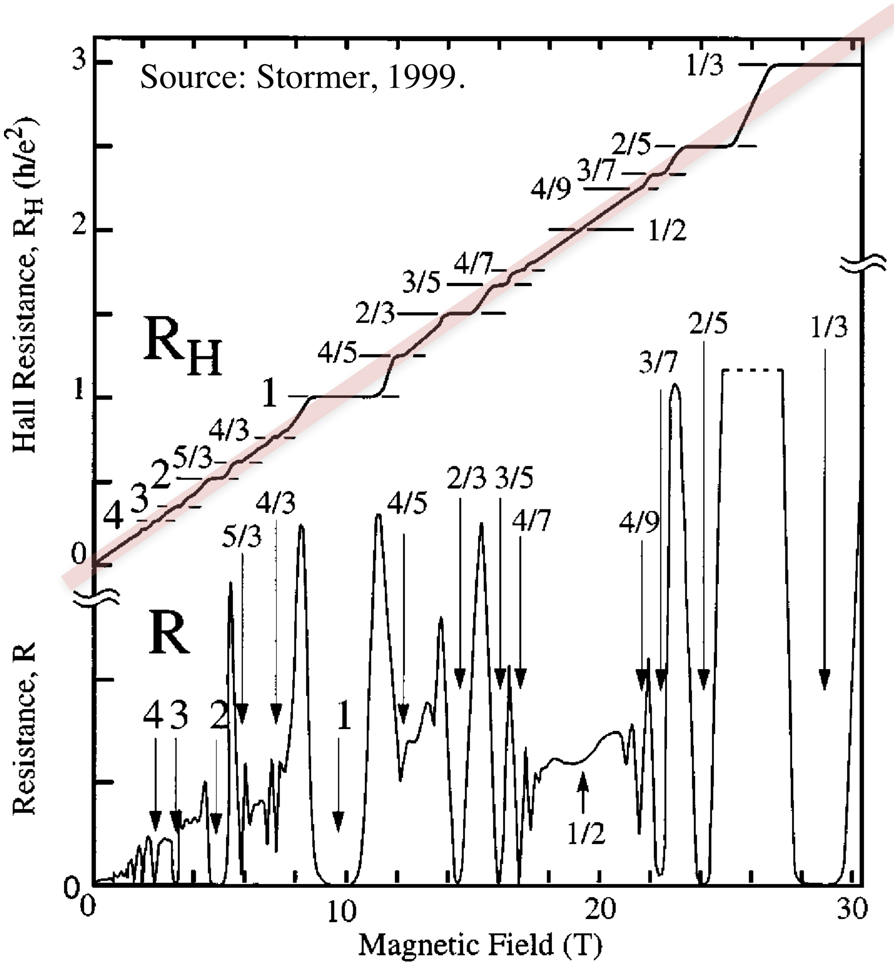
1) Transformation properties related to kinetic coefficients - clarifies and simplifies adiabatic transport arguments

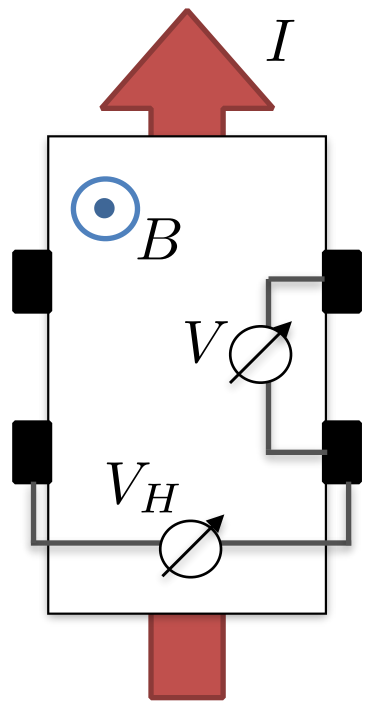
2) More structure is revealed when we allow inhomogeneous deformations of field and metric

3) Informs construction of effective field theories

Quantum Hall Effect: Historical Background

Classical electrodynamics –> $R_H = B/e\rho$





- Hall Resistance

$$R_H = V_H/I = \frac{1}{\nu}\frac{h}{e^2}$$

- Filling fraction

$$\nu = 2\pi l^2 \rho$$

Landau level degeneracy, 1 state per area $2\pi l^2$

- Gap to Excitations

- Magnetoresistance

$$R \sim e^{-\Delta_\nu / k_B T}$$

Integer filling

$$\Delta_\nu \approx \hbar\omega_c$$

separation of LLs

Rational filling

$$\Delta_\nu = e^2/\epsilon l$$

Coulomb interaction

Fractional Quantum Hall states: Laughlin wave function

Laughlin ground state wave function

$$\Psi_L = \prod_{i<j} (z_i - z_j)^\beta e^{-\frac{1}{4l^2}\sum_i |z_i|^2}$$

Mean density in **bulk:**

$$\langle \rho(z) \rangle = \frac{1}{\mathcal{Z}} \int \prod_{i=2}^{N} |\Psi_L(z, z_2, ..., z_N)|^2 = \frac{1}{2\pi\beta l^2}$$

Filling fraction: $\nu = 2\pi l^2 \rho = \frac{1}{\beta}$ $\sigma_H = \frac{e\rho}{B} = \nu \frac{e^2}{h}$

Incompressible Quantum Liquid

FQH liquid in “topological” sector (below gap) characterized by robust kinetic coefficients - ground state properties

– Hall conductance: E&M response

$$J_i = \sigma_H \epsilon_{ij} E_j$$

$$\boxed{\sigma_H = \nu \frac{e^2}{h}}$$

Laughlin ’81 (IQH) , ’83 (FQH)

– Odd Viscosity: gravitational response

Viscosity tensor

$$\Pi_{ij} = \sum_{k,l} \eta_{ijkl} \frac{\partial v^k}{\partial x^l}$$

In 2D isotropic liquid, odd part has one independent component

$$\eta^A_{ijkl} = \frac{1}{2}(\eta_{ijkl} - \eta_{klij})$$

$$\nabla_i \Pi^{ij} = \eta_A \epsilon^{jk} \nabla^2 v_k$$

$$\boxed{\eta_A = \frac{\hbar n \mathcal{S}}{4}}$$

IQH: Avron, Seiler, Zograf ’95
FQH: Tokatly, Vignale ‘07, ‘09
Read ’09
Read, Rezayi ‘11

$\mathcal{S}$ “shift”, takes integer values Wen & Zee ’92

Lowest Landau Level in a curved space

LLL annihilation operator:

$$\left(-i\hbar\frac{\partial}{\partial\bar{z}} - ie\bar{A}\right)\Psi = 0$$

$\boxed{\text{Gauge potential encodes metric}}$

$$ds^2 = \sqrt{g}(dx^2 + dy^2) \qquad \nabla\times\mathbf{A} = B\sqrt{g}$$

$$z = x + iy$$

Constant magnetic field, curved surface:

$$\Psi = F(\{z_i\})\exp\left[-\frac{1}{4l^2}\sum_i K(z_i,\bar{z}_i)\right]$$

Kahler potential:

$$\partial\bar{\partial}K = 2g_{z\bar{z}} = \sqrt{g}$$

[e.g. Sphere: $K = 4r^2\log(1+|z|^2/4r^2)$ Plane: $K = |z|^2$]

Weakly non-uniform magnetic field, curved surface:

$$\Psi = F(\{z_i\})\exp\left[-\frac{e}{\hbar}\sum_i \Phi(z_i,\bar{z}_i)\right]$$

$$\Delta_g\Phi = B(\mathbf{r})$$

Laplace-Beltrami

FQH model wave functions in LLL

The holomorphic factor F of the wave function on genus zero surfaces is the same as in the flat case. In this talk, I will focus on the Laughlin wave function, in which case

$$F(\{z_i\}) = \prod_{i<j} (z_i - z_j)^\beta \qquad \nu = \frac{1}{\beta}$$

Higher genus requires imposing periodic b.c., requires elliptic theta functions

Shifted Large N limit

Degeneracy of Lowest Landau Level (Riemann-Roch Theorem) $N_\phi + \frac{\chi}{2}$ $N_\phi = \frac{1}{\phi_0}\int BdA$

Ground state of Integer Hall effect constructed by completely filling LLL $N_1 = N_\phi + \frac{\chi}{2}$

Ground state of fractional quantum Hall effect experiences similar offset due to topology $N_\nu = \nu N_\phi + \nu \mathcal{S}\frac{\chi}{2}$

Wen, Zee '92

The setting for our problem:

$N = N_\nu$ number of particles so there is no boundary (no low-energy edge states), a constant magnetic field strength $B = l^{-2}$ (for now) and the limit

$l \to 0 \quad N_\phi \to \infty \qquad 2\pi l^2 N_\phi = A$ Area held constant

Generating Functional

- **Generating functional** (i.e. normalization or partition function)

$$\mathcal{Z}[W] = \int \prod_{i<j} |z_i - z_j|^{2\beta} \prod_{i=1}^{N} e^{W(z_i, \bar{z}_i)} d^2 z_i$$

1) constant B, curved space: $W = -\frac{1}{2l^2} K + \log \sqrt{g}$

2) non-uniform B, flat space: $W = -\frac{2e}{\hbar} \Phi + \log \sqrt{g}$

$$\frac{\delta \log \mathcal{Z}}{\delta W(z)} = \left\langle \sum_i \delta^{(2)}(z - z_i) \right\rangle = \langle \sqrt{g} \rho(z) \rangle \qquad \frac{\delta^2 \log \mathcal{Z}}{\delta W(z) \delta W(z')} = \langle \sqrt{g} \rho(z) \sqrt{g} \rho(z') \rangle_c$$

Generates connected correlation functions of density. Variations in W are connected to:
1) variations of metric, or 2) variations of magnetic field

Linear Response Relations

$$\frac{q^4}{2}\eta(q) = \left(1 + \frac{q^2}{2}\right) s(q) - \frac{q^2}{2} \qquad q = kl$$

$$q^2\sigma(q) = \nu\frac{e^2}{h}2s(q)$$

Definitions:

- $s(k) = \langle \rho_k \rho_{-k} \rangle_c$ Static structure factor
- $\sigma = \left.\frac{\delta\langle\rho\rangle}{\delta B}\right|_{B=B_0}$ Generalized Streda formula for Hall conductance.
- $\eta = \frac{1}{\langle\rho\rangle l^2}\left.\frac{\delta\langle\rho\rangle}{\delta R}\right|_{R=0}$ Density response to curvature

Odd Viscosity and Curvature Response

Odd viscosity appears in momentum dependent Hall conductance

$$\frac{\sigma(q)}{\sigma(0)} = 1 + q^2\left(\eta(0) - \frac{1}{2}\right) + \mathcal{O}(q^4)$$

cf. Hoyos & Son, 2012;
Bradlyn, Goldstein & Read, 2012

$$\sigma(0) = \nu\frac{e^2}{h}$$

Hall conductance

$$\eta(0) = \eta_A/\rho_0\hbar$$

$$\rho_0 = \nu/2\pi l^2$$

Curvature response at zero momentum identified with Odd viscosity. Contrast to adiabatic transport done on flat metrics.

Ward Identity for Laughlin States

- **Ward Identity** for Laughlin state: Generating functional invariant under coordinate transformation.

Consider $$z_i \to z_i + \frac{\epsilon}{z - z_i}$$

$$\int \prod_j d^2 z_j \sum_i \partial_i \left(\frac{1}{z - z_i} \prod_{k<l} |z_k - z_l|^{2\beta} e^{\sum_p W(z_p)} \right) = 0$$

$$\boxed{-2\beta \int \frac{\partial W(\xi)}{z - \xi} \langle \rho(\xi) \rangle \sqrt{g} d^2\xi = (2 - \beta)\langle \partial^2 \varphi(z) \rangle + \langle (\partial \varphi(z))^2 \rangle}$$

Zabrodin, Weigmann '06

$$\varphi(z) = -\beta \sum_i \log |z - z_i|^2 \qquad -\Delta_g \varphi(z) = 4\pi\beta\rho(z)$$

Exact equation connecting one- and two- point correlation functions

$$\langle (\partial \varphi(z))^2 \rangle = \lim_{z' \to z} \langle \partial \varphi(z) \partial \varphi(z') \rangle$$

To close hierarchy, need additional data

Asymptotic Expansion and Short Distance Regularization

- Asymptotic expansion: $l \to 0 \quad N_\phi \to \infty \quad 2\pi l^2 N_\phi = A$ constant

Green function of Laplace-Beltrami

$$\langle \varphi(z)\varphi(z')\rangle_c = \beta G(z,z') \to -\beta \log |z-z'|^2$$

Leading order in l, valid at large separation

- UV Regularization: covariant regularization of Green function:

$$\langle \varphi(z)\varphi(z')\rangle_c = \beta\left(G(z,z') + 2\log d(z,z')\right)$$

Missing ingredient for Ward identity

$$\boxed{\langle (\partial\varphi)^2\rangle_c = \frac{\beta}{6}\left[\partial^2 \log\sqrt{g} - \frac{1}{2}(\partial \log\sqrt{g})^2\right]}$$

Particle Density for Laughlin states: Asymptotic expansion

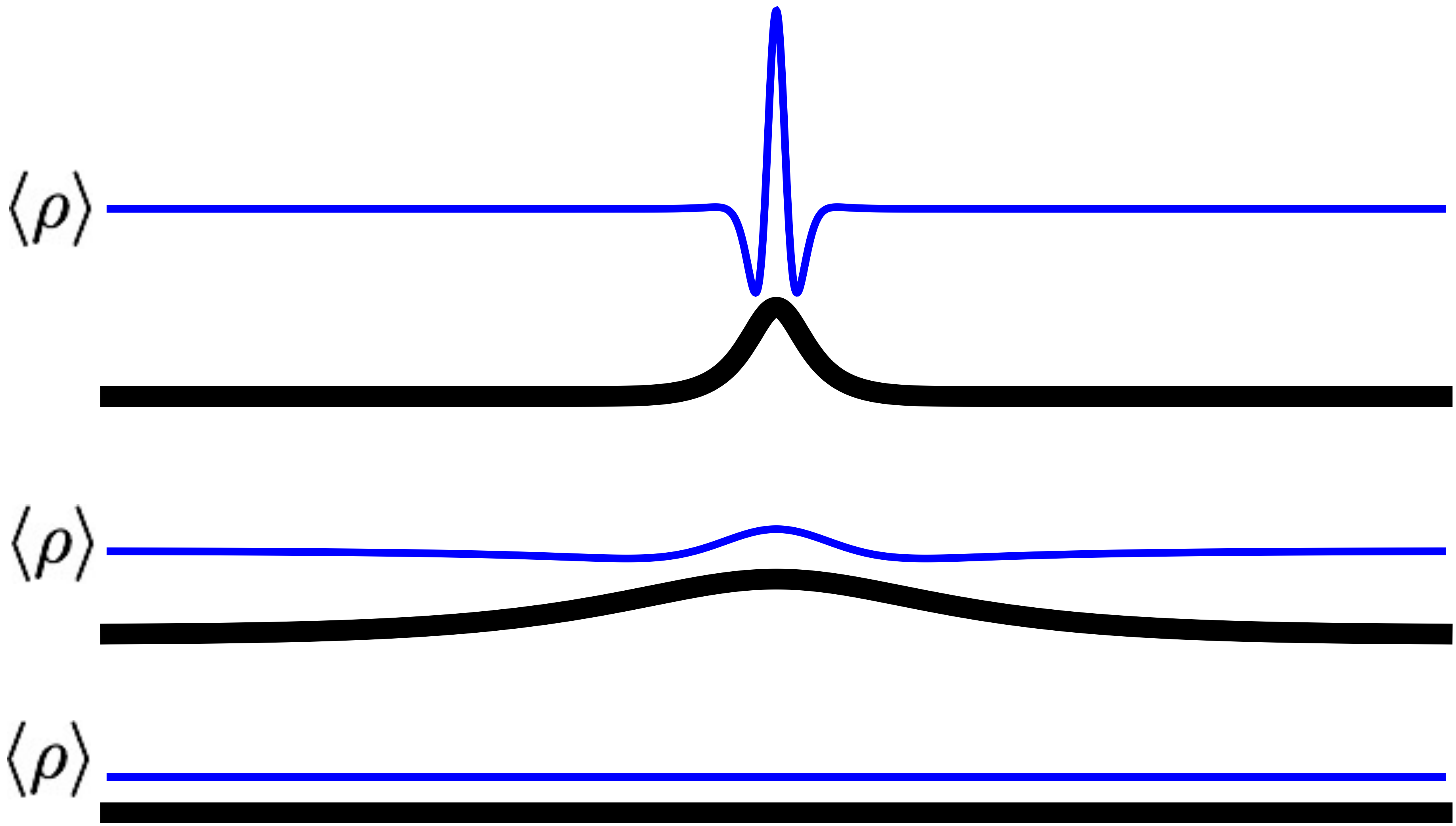
$$\boxed{\langle\rho\rangle = \frac{1}{2\pi\beta l^2} + \frac{1}{8\pi}R + \frac{1}{8\pi}\left(\frac{1}{12} + \frac{2-\beta}{4}\right) l^2\Delta_g R + O(l^4)}$$

- Scalar curvature: $R = -\Delta_g \log\sqrt{g}$
- Definition of "shift": $N = \nu N_\phi + \nu\mathcal{S}\frac{\chi}{2} \qquad \Rightarrow \mathcal{S} = \beta$

- Particle density is local, coordinate invariant quantity => composed entirely of curvature invariants: Scalar curvature and Laplace-Beltrami. Functional form is **FIXED**, only coefficients may vary. True for any state in LLL.

$\beta = 1$ -Asymptotic expansion of Bergman Kernel of Yau-Tian-Zelditch
-Free fermions filling LLL: Douglas & Klevtsov 2010

Density Profiles on Curved Surface



profile of cylindrical surface
(think boa constrictor after a meal)

Particle Density for Laughlin states: Asymptotic expansion

- Curvature Response: $\eta = (\rho_0 l^2)^{-1} \delta\langle\rho\rangle / \delta R$

$$\eta(q) = \frac{\beta}{4} - \frac{\beta}{4}\left(\frac{1}{12} + \frac{2-\beta}{4}\right) q^2 + O(q^4) \qquad \eta_A = \hbar\rho_0\eta(0)$$

- Static Structure Factor determined up to sixth moment

$$s(q) = \frac{q^2}{2} + \frac{\beta-2}{8} q^4 - \frac{\beta}{96}\left(1 - \frac{3(2-\beta)^2}{\beta}\right) q^6 + \ldots$$

  - The sixth moment (i.e. q^6 coefficient) is currently state of the art - found in 2000 by Kalinay et al. Unclear what the analytical structure of higher moments are.

Log-Gas Picture

Partition function:

$$\mathcal{Z} = \int \prod_{i=1}^{N} d^2 z_i e^{-\beta E}$$

Energy:

$$E = -\sum_{i<j} \log |z_i - z_j|^2 + \frac{1}{2\beta l^2} \sum_i |z_i|^2$$

Pair Distribution function:

$$\rho(1)\rho(2)g(1,2) = \frac{N(N-1)}{\mathcal{Z}} \int \prod_{i=3}^{N} d^2 z_i e^{-\beta E}$$

FQH on a Potato

$$\langle\rho\rangle = \rho_0 + \rho_1 R + \rho_2 \Delta_g R + \rho_3 \Delta_g^2 R + \ldots$$

(ignoring terms nonlinear in R)

$$\frac{q^4}{2}\eta(q) = \left(1 + \frac{q^2}{2}\right) s(q) - \frac{q^2}{2}$$

Sum Rules for Log-Gas

$$\rho \int r^{2n}(g(r) - 1) d^2\mathbf{r}$$

Particle Density: Non-Uniform Magnetic Field

- Particle Density

$$\langle \rho \rangle = \frac{\nu}{2\pi} B + \frac{(2-\beta)}{8\pi\beta} \Delta \log B - \frac{1}{96\pi} \left(1 - 6\frac{(2-\beta)^2}{2\beta}\right) \Delta \left(\frac{1}{B} \Delta \log B\right)$$

- Hall Conductance (Response to Magnetic Field): $\sigma_H(q) = \frac{\delta\langle\rho\rangle}{\delta B}\Big|_{B=B_0}$

$$\sigma_H(q) = \frac{\nu}{2\pi}\left\{1 + \frac{\beta-2}{4}q^2 - \frac{\beta}{48}\left(1 - \frac{3(2-\beta)^2}{\beta}\right)q^4\right\} + O(q^6)$$

- Prediction for Laughlin states ONLY - in general will receive corrections from diamagnetic currents at q^2

-Could have actually gotten this directly from the curvature response - not an independent quantity!

$$q^2\eta(q) + 1 = \left(1 + \frac{q^2}{2}\right)\beta\sigma_H(q)$$

Generating Functional

- By integrating identity: $$-\frac{1}{2}l^2\Delta_g \frac{\delta \log \mathcal{Z}[g]}{\delta\sqrt{g}} = (1 - \frac{1}{2}l^2\Delta_g)\langle\rho\rangle$$

$$\log\frac{\mathcal{Z}[g]}{\mathcal{Z}[g_0]} = \frac{N_\phi}{2}(N_\nu + 1) + N_\phi^2 A^{(2)}[g] + N_\phi A^{(1)}[g] + A^{(0)}[g]$$

$$A^{(2)} = -\frac{\pi}{2\beta}\frac{1}{V^2}\int K\,dV \qquad A^{(1)} = \frac{1}{2V}\int \log\sqrt{g}\,dV$$

$$A^{(0)} = \left(\frac{1}{24\pi} + \frac{\beta-1}{16\pi}\right)\left(\frac{1}{2}\int \log\sqrt{g}R\,dV + 8\pi\right)$$ Liouville Action

- Higher order terms believed to be exact one-cocycles: for $g = e^\phi g_0$ action functional takes the form of a difference $A[g] - A[g_0]$ of *local* functionals of the metric. This is not so with the leading three terms.

Partial Results for Pfaffian State

Pfaffian wave functions describes fermions at even denominator

$$F(\{z_i\}) = \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)^{\beta} \qquad \nu = \frac{1}{\beta}$$

$$\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) = \mathcal{A}\left(\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\dots\frac{1}{z_{N-1} - z_N}\right) = \langle\psi(z_1)\dots\psi(z_N)\rangle$$

N-point correlator of free Majorana fermions

Partial results for Pfaffian State

$$\langle\rho\rangle = \frac{\nu}{2\pi l^2} + \frac{\nu\mathcal{S}}{8\pi}R + \frac{1}{8\pi}\left(\frac{1+c_\psi}{12} + \frac{\nu\mathcal{S}(2-\mathcal{S})}{4}\right) l^2\Delta_g R$$

$$\eta(q) = \frac{\mathcal{S}}{4} - \frac{\mathcal{S}}{4}\left(\frac{1+c_\psi}{12} + \frac{\nu\mathcal{S}(2-\mathcal{S})}{4}\right) q^2 + \mathcal{O}(q^4)$$

$$s(q) = \frac{q^2}{2} + \frac{\mathcal{S}-2}{8}q^2 - \frac{1}{96\nu}\left(1 + c_\psi - 3\nu(2-\mathcal{S})^2\right) q^6 + \mathcal{O}(q^8)$$

Numerically accessible

Odd viscosity

$$\eta_A = \hbar\rho_0\eta(0) = \frac{\hbar\rho_0\mathcal{S}}{4}$$

$$\mathcal{S} = \nu^{-1} + 1$$

$c_\psi = 1/2$ Majorana central charge

$1 + c_\psi$ Thermal Hall coeff. ?

Why do these coefficients matter?

- Are they universal? Do they receive corrections in realistic systems?

- **Hypothesis:** They fix terms in the geometrical and topological effective action, which is robust in a given FQH phase. These can be used to define what is meant by a FQH phase.

Geometrical Action for FQH effect

$$S = \frac{\sigma_H}{2}\int AdA + \frac{\nu\mathcal{S}}{4\pi}\int Ad\omega + C\int \omega d\omega$$

A Gauge potential

ω spin connection

For Laughlin states:

$$C = \frac{1}{24\pi} + \frac{\beta - 1}{16\pi}$$

Gromov, Abanov '14

$$A^{(0)} = \left(\frac{1}{24\pi} + \frac{\beta-1}{16\pi}\right)\left(\frac{1}{2}\int \log\sqrt{g}R\, dV + 8\pi\right)$$

This **coincidence** suggests the geometrical action can be recovered from the wave function as an **Adiabatic (Berry) phase**.

Which would imply the gravitational sector of the effective theory is accessible from ground state