Deformed Ginibre ensembles and integrable systems

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Abstract

We consider three Ginibre ensembles (real, complex and quaternion-real) with deformed measures and relate them to known integrable systems by presenting partition functions of these ensembles in form of fermionic expectation values. We also introduce double deformed Dyson-Wigner ensembles and compare their fermionic representations with those of Ginibre ensembles.

Key words: integrable systems, random matrices, matrix models, tau functions, Pfaffians, BKP, DKP, two-component Toda lattice, free fermions, double deformed Dyson-Wigner ensembles, double deformed Ginibre ensembles.

1 Introduction

Ginibre ensembles play an important role in many physical and statistical problems. A list of several applications of these ensembles has been presented in the review article [1] and includes problems in quantum chromodynamics, dissipative quantum maps, scattering in chaotic quantum systems, growth processes, fractional quantum-Hall effect, Coulomb plasma, stability of complex biological and neural networks, directed quantum chaos in randomly pinned superconducting vortices, delayed time series in financial markets, and random operations in quantum information theory.

They were introduced in 1965 by Ginibre as non-Hermitian analogues of the famous Wigner-Dyson ensembles. Dyson-Wigner ensembles consist of random real symmetric, Hermitian and quaternionic self-dual random matrices, and thanks to their symmetry classes, are also known as orthogonal (OE), unitary (UE) and symplectic (SE) Wigner-Dyson ensembles. The non-Hermitian analogues are called real Ginibre, complex Ginibre and quaternion-real Ginibre ensembles often referred to as GinOE, GinUE and GinSE respectively.

Let us recall that both Wigner-Dyson ensembles and their Ginibre counterparts are ensembles with Gaussian probability measure[1]. This is enough for a number of problems which are considered by physicists working in quantum chaos. However, for many other applications of random matrices different measures obtained as deformations of the Gaussian one may be of use. It is well known that certain deformations of the Gaussian measure of Wigner-Dyson ensembles make it possible to identify partition functions of these ensembles with tau functions of integrable hierarchies where deformation parameters are treated as so-called higher times. At first this connection was established for the unitary ensemble which was identified with a 1D Toda lattice tau function [13]. Later, it was also established also for the orthogonal and symplectic ensembles which were related to the Pfaff lattice in [7], and in [5] also to a more general "large BKP hierarchy" introduced by Kac and van de

[1] there has been a recent resurgence in research into Ginibre ensembles (e.g. the discovery of the Pfaffian structure of eigenvalue correlation functions for real Ginibre by Borodin and Sinclair [34], Nagao and Forrester [35], G. Akemann and E. Kanzieper [28]; the discovery of the connection between coalescing (annihilating) Brownian motions on $\mathbb{R}$ and the real Ginibre ensemble by Tribe and Zaboronski [36]. All these studies deal with Gaussian model potentials.

Leur in [4]. Besides, in [10] orthogonal and symplectic ensembles of even-size matrices were also related to a so-called coupled KP equation (which is actually a part of the DKP hierarchy introduced in [9]).

The link between the real and the quaternion-real Ginibre ensembles and integrable systems has remained unknown. Below I shall fill this gap. This supplements the picture of an interesting interplay between integrable models and various models describing statistics of random behavior (sometimes called models of integrable probability).

I will show that the partition functions of the real and quaternion-real Ginibre ensembles may be viewed as tau functions, like in many other models of random matrices [13], [14], [22], [24], [7], [5], [10], [25]. Here we use the BKP hierarchy introduced in [4] (named by the authors "charged" or "fermionic" BKP while we prefer to call it large BKP) and the large 2-BKP hierarchy introduced in [3]. The large 2-BKP hierarchy is rather similar to the 2-DKP (Pfaff-DKP) one introduced in [21] [2]. Let us note that the BKP tau function depends on two discrete numbers, N and L, and a semi-infinite set of complex numbers $\mathbf{t} = (t_1, t_2, \ldots)$, whilst the 2-BKP tau function depends also on an additional semi-infinite set of complex parameters $\mathbf{s} = (s_1, s_2, \ldots)$. These sets of numbers and parameters are called higher times of the corresponding hierarchy, see Section 2 below.

We shall see that the real and quaternion-real Ginibre ensembles, both deformed in an appropriate way, can be related to the BKP hierarchy. The discrete variable N represents the size of the matrices from a given ensemble, while the parameters $L \geq 0$ and $\mathbf{t}$ describe the deviation of the probability measure from the Gaussian one. We shall refer to such ensembles as deformed Ginibre ensembles (d-GinOE, d-GinUE and d-GinSE). It will be proven that the partition functions of d-GinOE and of d-GinSE are BKP tau functions. The remaining case of d-GinUE is the easiest one because the d-GinUE partition function basically coincides with the partition function of the model of normal matrices and, therefore, may be related to the two-component KP hierarchy as presented in [18].

If we restrict Ginibre ensembles by the additional requirement of restricting our ensembles to ensembles of invertible matrices (we call such ensembles *restricted* Ginibre ensembles), then we can introduce more general deformations described by an integer L and two sets $\mathbf{t}$ and $\mathbf{s}$. We refer to these ensembles as double deformed ones and denote them as dd-GinOE, dd-GinUE and dd-GinSE. Then the partition functions of dd-GinOE and of dd-GinSE coincide with tau functions of the 2-BKP hierarchy. The double deformed complex Ginibre ensemble, dd-GinUE, may be related to the two-component Toda lattice.

A fermionic representation for the tau functions will be written down and compared with the fermionic representation of the similarly deformed classical Dyson-Wigner ensembles, where the additional set of parameters $\mathbf{s}$ is included.

2 Gaussian Real Ginibre ensemble and its deformation

Real Hermitian ensemble. First, let us recall that the ensemble of real Hermitian matrices (also known as the Dyson-Wigner orthogonal ensemble) is given by the following partition function

$$I_N^{d-OE}(\mathbf{t}) = \int d\mu(X, \mathbf{t}) \,, \quad d\mu(X, \mathbf{t}) = e^{-\frac{1}{2}\mathrm{tr}(X^2) + \mathrm{tr} V(X, \mathbf{t})} \prod_{i \geq j} dX_{ij} \tag{1}$$

$$V(x, \mathbf{t}) := \sum_{n=1}^{\infty} x^n t_n \tag{2}$$

[2]The large BKP hierarchy includes the famous KP one as a particular reduction, while the large 2-BKP includes the Toda lattice hierarchy [20]. In the present text we will refer to these hierarchies as BKP and 2-BKP ones instead of the "large BKP" and "large 2-BKP". One must not confuse the "large" BKP hierarchy which contains the KP one and the "small" BKP hierarchy (referred to as "neutral" in [4]) introduced in [30] and which is contained in the KP hierarchy.

where X is a real $N\times N$ Hermitian matrix and $\mathbf{t}=(t_1,t_2,\dots)$ is a set of parameters which describes the deviation of the probability measure from the Gaussian one.

Let us restrict the space of our symmetric matrices to invertible symmetric matrices. Then a more general deformation of the measure may be considered as follows

$$I_N^{dd-OE}(L,\mathbf{t},\mathbf{s},\Pi)=\int d\mu(X,L,\mathbf{t},\mathbf{s},\Pi)\,,\qquad d\mu(X,L,\mathbf{t},\mathbf{s},\Pi)=\mathrm{det}X^L e^{-\mathrm{tr}V(X^{-1},\mathbf{s})}\,\Pi(X)\,d\mu(X,\mathbf{t}) \quad (3)$$

where $\mathbf{s}$ and an integer L is a collection of new deformation parameters. Here the factor $\Pi(X)$ has the form

$$\Pi(X)=\prod_{i=1}^N \pi(x_i)$$

where $x_i,\ i=1,\dots,N$ are the eigenvalues of X, and the function π may be chosen in an arbitrary fashion, provided the integral (3) converges.

Dyson-Wigner Gaussian ensemble will be referred to as G-OE, while ensembles (1) and (3) will be referred to respectively as d-OE and dd-OE.

It was found by M.Adler and P. van Moerbeke in [7] that for even N the ensemble (1) where $\Pi(X)=1$ may be related to the Pfaff Toda lattice [8]. A little bit later in [5] J. van de Leur found that, for both even and odd N, and for arbitrary admissible π, the orthogonal ensemble (1) may be also identified with the (large) BKP tau function introduced in [4]. Below we shall identify the ensemble (3) with the (large) 2-BKP tau function introduced in [3].

In terms of the eigenvalues $x_1,\dots,x_N$ of the Hermitian matrix X, the integral (3) may be written as

$$I_N^{dd-OE}(L,\mathbf{t},\mathbf{s},\Pi)=a_N\int_{x_1>\cdots>x_N}\Delta(x_1,\dots,x_N)\prod_{i=1}^N x_i^L e^{-\frac12 x_i^2+V(x_i,\mathbf{t})-V(x_i^{-1},\mathbf{s})}\pi(x_i)dx_i \quad (4)$$

with some constant a_N related to the volume of the orthogonal group $O(N)$. Deformation parameters L, $\mathbf{t}$ and $\mathbf{s}$ are considered to be chosen in such a way that the integral (4) is convergent.

Real non-Hermitian ensemble. Let us turn to the Ginibre ensembles. The Gaussian real Ginibre ensemble, also known as the Gaussian Ginibre orthogonal ensemble (G-GinOE) is defined on the space of real matrices by assigning to each entry a Gaussian probability measure with constant variance:

$$I_N^{G-GinOE}=\int d\mu(X)\,,\quad d\mu(X)=\prod_{i,j}e^{-\frac12 X_{ij}^2}dX_{ij}=e^{-\frac12\mathrm{tr}(XX^\dagger)}\prod_{i,j}dX_{ij} \quad (5)$$

where X is $N\times N$ matrix with real entries. The measure defined in (5) is invariant under orthogonal transformations of matrices X.

The so-called elliptic deformation of the G-GinOE measure,

$$d\mu(X,a)=e^{-a\mathrm{tr}X^2}d\mu(X), \quad (6)$$

with deformation parameter a, was studied in [1] [3] .

We shall consider the following multi-parameter deformation of the G-GinOE ensemble (below referred to as d-GinOE)

$$I_N^{d-GinOE}(L,\mathbf{t},\Pi)=\int d\mu(X,L,\mathbf{t},\Pi)\,,\qquad d\mu(X,L,\mathbf{t},\Pi)=\Pi(X)\mathrm{det}X^L\,e^{\mathrm{tr}V(X,\mathbf{t})}\,d\mu(X) \quad (7)$$

[3] It is also called the elliptic deformation of G-GinOE because, in the large N limit, the parameter a describes a deformation of a circle equilibrium domain of the eigenvalues in complex plane to an elliptic form.

where V is given by (2) and where $L = 0, 1, 2 \dots$[4] and sets of numbers $\mathbf{t} = (t_1, t_2, \dots)$ are deformation parameters. The factor $\Pi(X)$ is a function of the eigenvalues of X. If a real $N \times N$ matrix X has $2m$ complex conjugate eigenvalues $z_i, \bar{z}_i, i = 1, \dots, m$ and $N - 2m$ real eigenvalues $x_i, i = 2m+1, \dots, N$ where $m = 0, \dots, [\frac{N}{2}]$ then

$$\Pi(X) = \prod_{i=1}^{m} \pi(z_i, \bar{z}_i) \prod_{i=2m+1}^{N} \pi(x_i) \,. \tag{8}$$

Functions $\pi(z, \bar{z})$ and $\pi(x)$ are not necessarily continuous. To keep the probability interpretation we ask $\Pi(X)$ to be non-negative but it is not important for us. We assume that $\mathbf{t}$ and Π are chosen in such a way that the partition function $I_N^{d-GinOE}(L, \mathbf{t})$ is finite.

Remark 1. Even in the case $\Pi = 1$ where in general the integral (7) is divergent, logarithmic derivatives with respect to the deformation parameters may be finite in the large N limit which is of physical interest. How this works for models of normal matrices is explained in [6].

Now let us consider the restricted Ginibre ensemble consisting of real invertible matrices, and deform its measure as follows

$$I_N^{dd-GinOE}(L, \mathbf{t}, \mathbf{s}, \Pi) = \int d\mu(X, L, \mathbf{t}, \mathbf{s}, \Pi) \,, \quad d\mu(X, L, \mathbf{t}, \mathbf{s}, \Pi) = \Pi(X) \, \mathrm{det} X^L \, e^{\mathrm{tr} V(X,\mathbf{t}) - \mathrm{tr} V(X^{-1}, \mathbf{s})} \, d\mu(X) \tag{9}$$

where X is a real invertible $N \times N$ matrix and V is given by (2), where an integer L and sets of numbers $\mathbf{t} = (t_1, t_2, \dots)$, $\mathbf{s} = (s_1, s_2, \dots)$ are deformation parameters. This ensemble will be called double deformed restricted Ginibre orthogonal ensemble (dd-GinOE). Again we suppose that the deformation parameters and the function π are chosen in a way which provides the existence of the integral $I_N^{dd-GinOE}(L, \mathbf{t}, \mathbf{s}, \Pi)$. Let us consider two examples.

Example 1.

For a polynomial choice of potentials $V(X, \mathbf{t})$ and $V(X^{-1}, \mathbf{s})$ (which means that only a finite number of parameters t_m and s_m is non-vanishing, see (2)), we may choose π to be vanishing for $|x| < r$, for some r small enough (infra-red cut-off) and for $|x| > R$, for some R large enough (ultra-violet cut-off). For instance, one may choose

$$\pi(x) = \begin{cases} 1 & \text{if} \quad r < |x| < R \\ 0 & \qquad \text{otherwise} \end{cases} \tag{10}$$

This allows one to avoid discussion of the problem of the existence of the integral (3). It may also be shown that the integral (3) does not depend on r or R for small enough r and for large enough R.

Example 2. If one takes $\mathbf{t} \to \mathbf{t} - \sum_{i=1}^{N_1} a_i [p_i]$ and $\mathbf{s} \to \mathbf{s} - \sum_{i=1}^{N_1} b_i [q_i]$, namely

$$t_n \to t_n - \frac{1}{n} \sum_{i=1}^{N_1} a_i p_i^n, \qquad s_n \to s_n - \frac{1}{n} \sum_{j}^{N_2} b_j q_j^n \tag{11}$$

where a_i, p_i and b_i, q_j are complex numbers, sometimes called Miwa variables, then the integral (9) reads

$$\int \prod_{i}^{N_1} \det \left(1 - p_i X\right)^{a_i} \prod_{j}^{N_2} \det \left(1 - q_i X^{-1}\right)^{-b_i} \, d\mu(X, L, \mathbf{t}, \mathbf{s}, \Pi) \tag{12}$$

In this example for $a_i, -b_j \geq 0$ one may chose $\pi = 1$.

It may be shown that any choice of the function π is compatible with (formal) integrability of the related ensemble, which means that it may be equated to a certain tau function.

[4]This parameter was recently introduced in [33] and relates to the so-called induced ensembles.

Now, let us prove that the integrals (7) and (9) may be identified with tau functions of the BKP hierarchy [4] and of the 2-BKP one [3] respectively. Then the size N of the matrices and the sets $L, \mathbf{t}, \mathbf{s}$ play the role of the higher times of this integrable hierarchy.

To make this identification, we should re-write integral (9) as an integral over eigenvalues. Let us recall that for the Gaussian real Ginibre ensemble (namely, for the case $\mathbf{t} = \mathbf{s} = 0$, $L = 0$) this problem was solved in [2], see also the review article [1]. This was done as follows. After applying the Schur decomposition $X = U(\Lambda + \Delta)U^\dagger$, where U is orthogonal, Λ is block-diagonal (where there are $\left[\frac{N}{2}\right]$ 2×2 blocks and an additional 1×1 block in case N is odd) and where Δ has nonzero blocks only above Λ , one arrives at

$$d\mu(X) = e^{-2\mathrm{Tr}(\Delta\Delta^\dagger + \Lambda\Lambda^\dagger)}|D\Delta|\,|D\Lambda|\,|{\prod}'\left(U^{-1}dU\right)_{ij}\frac{(\lambda_i - \lambda_j)}{2\pi}|$$

with the dashed product running over nonzero entries in the lower triangle of $U^\dagger dU$. The set of eigenvalues λ_i, $i = 1, \ldots, N$ of the real matrix X consists of a number, say k, of complex conjugated pairs, $z_1, \bar{z}_1, \ldots, z_k, \bar{z}_k$, such that $2k \leq N$, and of some number of real eigenvalues $x_1, \ldots, x_{N-2k}$. After integration over matrices Δ and U, one obtains an integral over eigenvalues only made up of integrals over real values x_i and over complex eigenvalues $z_j, \bar{z}_j$. Finally, after series of computations [2], the Gaussian real Ginibre ensemble may be written as the following integral over complex and over real eigenvalues:

$$I_N^{G-GinOE} = b_N \cdot \sum_{k=0}^{\left[\frac{N}{2}\right]} \int_{\mathbb{M}_{2k,N-2k}} \Delta_{2n}(z_1, \bar{z}_1, \ldots, z_k, \bar{z}_k, x_1, \ldots, x_{N-2k})\, d\Omega_{2k}^C\, d\Omega_{N-2k}^R \tag{13}$$

where the integration domain $\mathbb{M}_{2m,N-2m}$ is as follows: $\Re z_1 > \cdots > \Re z_m$, $x_{2m+1} > \cdots > x_N$, $z_i \in \mathbb{C}_+$ (upper half-plane), $x_i \in \mathbb{R}$, and where

$$d\Omega_{2m}^C(\mathbf{z}, \bar{\mathbf{z}}) = \prod_{i=1}^m \mathrm{erfc}\left(\frac{|z_i - \bar{z}_i|}{\sqrt{2}}\right) e^{-\Re z_i^2} d^2 z_i\,, \qquad d\Omega_{N-2m}^R(\mathbf{x}) = \prod_{i=2m+1}^N e^{-\frac{1}{2}x_i^2} dx_i \tag{14}$$

with $\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-x^2}$ (this function appears as a result of Gaussian integration of antisymmetric parts of 2 by 2 blocks of Λ where one takes into account that Λ is real, see [2]). Here the factor c_N absorbs integrals over Δ and over U. Let us note that the Gaussian case is a special case of (9): $I_N^{G-GinOE} = I_N^{dd-GinOE}(0,0,0)$. The factor b_N is independent of $L, \mathbf{t}, \mathbf{s}$ and was evaluated for GinOE in [2].

Now we notice that the deformation (9) results in multiplication of the measure $d\mu(X)$ by a factor which depends only on eigenvalues $\lambda_1, \ldots, \lambda_N$ of the matrix X:

$$d\mu(X) \to d\mu(X, \mathbf{t}, L, \mathbf{s}, \Pi) = d\mu(X)\,\Pi(X) \prod_{i=1}^N \lambda_i^L e^{V(\lambda_i, \mathbf{t}) - V(\lambda_i^{-1}, \mathbf{s})} \tag{15}$$

where V is given by (2). Then we introduce

$$d\Omega_{2m}^C \to d\Omega_{2m}^C(\mathbf{t}, L, \mathbf{s}, \Pi) = \prod_{i=1}^m \mathrm{erfc}\left(\frac{|z_i - \bar{z}_i|}{\sqrt{2}}\right) |z_i|^{2L} e^{2\Re V(z_i, \mathbf{t}) - 2\Re V(z_i^{-1}, \mathbf{s}) - \Re z_i^2} \pi(z_i, \bar{z}_i) d^2 z_i \tag{16}$$

$$d\Omega_{N-2m}^R \to d\Omega_{N-2m}^R(\mathbf{t}, L, \mathbf{s}, \Pi) = \prod_{i=2m+1}^N x_i^L e^{V(x_i, \mathbf{t}) - V(x_i^{-1}, \mathbf{s}) - \frac{1}{2}x_i^2} \pi(x_i) dx_i \tag{17}$$

where, at last, we specified $\Pi(X)$, $\Pi(X) = \prod_{i=1}^m \pi(z_i, \bar{z}_i) \prod_{i=2m+1}^N \pi(x_i)$. For a polynomial potential, one may take

$$\pi(x) = \begin{cases} 1 & \text{if} \quad r < |x| < R \\ 0 & \quad\text{otherwise} \end{cases}, \qquad \pi(z, \bar{z}) = \begin{cases} 1 & \text{if} \quad r < |z| < R \\ 0 & \quad\text{otherwise} \end{cases} \tag{18}$$

Alternatively, one may choose, say, $\pi(z,\bar z)=e^{-|\frac{z}{R}|^n-|\frac{r}{z}|^m}$ with large enough n and m.

Let us introduce

$$J_N(L,\mathbf{t},\mathbf{s},\Pi;\alpha)=$$
$$\sum_{k=0}^{[\frac{N}{2}]}\alpha^k\int_{\mathbb{M}_{2k,N-2k}}|z_i|^{2L}\Delta_{2n}(z_1,\bar z_1,\dots,z_k,\bar z_k,x_1,\dots,x_{N-2k})\,d\Omega^C_{2k}(\mathbf{t},L,\mathbf{s},\Pi)\,d\Omega^R_{N-2k}(\mathbf{t},L,\mathbf{s},\Pi) \quad (19)$$

We note that for $\alpha\to 0$ (up to a constant independent of the variables $L,\mathbf{t},\mathbf{s}$), this expression is equal to the partition function of the deformed Dyson-Wigner orthogonal ensemble (7). When $\alpha=1$, this is the partition function for the deformed real Ginibre ensemble (7).

$$J_N(L,\mathbf{t},\mathbf{s}=0,\Pi;\alpha=1)=b_N I_N^{d-GinOE}(L,\mathbf{t},\Pi) \quad (20)$$

$$J_N(L,\mathbf{t},\mathbf{s},\Pi;\alpha=0)=a_N I_N^{dd-OE}(L,\mathbf{t},\mathbf{s},\Pi)\,,\qquad J_N(L,\mathbf{t},\mathbf{s},\Pi;\alpha=1)=b_N I_N^{dd-GinOE}(L,\mathbf{t},\mathbf{s},\Pi) \quad (21)$$

The factors a_N and b_N are independent of deformation parameters and may be found respectively in [11] and [1].

In case of polynomial V, we can introduce the cut-off as in (10). Based on standard studies of random matrices, see [6], it is natural to suppose that, in this case, in the large N limit, the integral (19) does not depend on r for small enough r, and does not depend on R for large enough R, however we leave this to one side as a separate mathematical problem.

Tau functions and matrix ensembles. Let us recall the fermionic construction of tau functions [9]. That is, here we present a general fermionic expression for the 2-BKP tau function and relate it to ensembles (3) and (9). In the next section it will also be related to a deformed symplectic Dyson-Wigner ensemble and to a deformed quaternion-real Ginibre ensemble.

The very notion of tau function and its fermionic construction was introduced by the Kyoto school. Here we use it in a version suggested in [4] where an additional fermionic mode ϕ was added, see Appendix A.2.

Following [9] we consider

$$\Gamma_+(\mathbf{t})=\exp\sum_{n=1}^\infty t_n\sum_{i\in\mathbb{Z}}\psi_i\psi^\dagger_{i+n},\qquad \Gamma_-(\mathbf{s})=\exp\sum_{n=1}^\infty s_n\sum_{i\in\mathbb{Z}}\psi_i\psi^\dagger_{i-n} \quad (22)$$

We shall need the following equality

$$\Gamma_+(\mathbf{t})\psi(z)\Gamma_-(\mathbf{s})=c(\mathbf{t},\mathbf{s})e^{V(z,\mathbf{t})-V(z^{-1},\mathbf{s})}\Gamma_-(\mathbf{s})\psi(z)\Gamma_+(\mathbf{t}) \quad (23)$$

where

$$c(\mathbf{t},\mathbf{s})=\exp\sum_{n=1}^\infty nt_ns_n \quad (24)$$

The 2-BKP tau function may be presented in the form of the following fermionic expectation value

$$\tau_N(L,\mathbf{t},\mathbf{s})=\langle N+L|\,\Gamma_+(\mathbf{t})\,e^{\Phi}\,\Gamma_-(\mathbf{s})\,|L\rangle \quad (25)$$

where Φ is any quadratic expression in the fermionic modes $\psi_i,\psi_i^\dagger,\phi$, see Appendix A.4. The variables $N,L,\mathbf{t},\mathbf{s}$ play the role of 2-BKP higher times. The tau function (25) solves the 2-BKP Hirota equations, see [3] and Appendix A.6.

Remark 2. The 2-BKP tau function is also a BKP tau function [4] with respect to the variables $N,L,\mathbf{t}$, as well as a BKP tau function with respect to the variables $N,L,\mathbf{s}$ (this explains the name 2-BKP).

The tau function (25) contains as particular cases: tau functions of the Toda lattice hierarchy [9], [20], the charged BKP tau function [4] and the 2-DKP ("Pfaff DKP") tau function [21]. Each of these cases may be obtained as special cases of the tau function (25), namely, by specifying Φ and $\mathbf{s}$, see Appendix A.4.

Theorem 1. *We have*

$$(-)^{NL}c(\mathbf{t},\mathbf{s})J_N(L,\mathbf{t},\mathbf{s};\alpha) = \langle N+L|\,\Gamma_+(\mathbf{t})\,e^{\alpha\Phi_c}e^{\Phi_r}\,\Gamma_-(\mathbf{s})\,|L\rangle \tag{26}$$

where

$$\Phi_c = \int_{\mathbb{C}_+} \mathit{erfc}\left(\frac{|z-\bar z|}{\sqrt{2}}\right)\psi(z)\psi(\bar z)e^{-\Re z^2}\pi(z,\bar z)d^2z \tag{27}$$

is an integral over the upper half of the complex plane and where

$$\Phi_r = \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{sgn}(x_1-x_2)\psi(x_1)\psi(x_2)e^{-\frac{1}{2}x_1^2-\frac{1}{2}x_2^2}\pi(x_1)\pi(x_2)dx_1dx_2 + \sqrt{2}\int_{\mathbb{R}}\psi(x)\phi e^{-\frac{1}{2}x^2}\pi(x)dx \tag{28}$$

Therefore thanks to (20) and to Remark 2, $(-1)^{NL}I_N^{d-GinOE}(L,\mathbf{t},\Pi)$ is a tau function of the large BKP, while, thanks to (21), both $(-1)^{NL}c(\mathbf{t},\mathbf{s})I_N^{dd-OE}(L,\mathbf{t},\mathbf{s},\Pi)$ and $(-1)^{NL}c(\mathbf{t},\mathbf{s})I_N^{dd-GinOE}(L,\mathbf{t},\mathbf{s},\Pi)$ are tau functions of the large 2-BKP ones.

Here is a sketch of the proof of (26). We transform the vacuum expectation value in the right-hand side to get the integral in the left-hand side as follows. First, we use the Taylor expansion of the exponentials $e^{\alpha\Phi_c}$ and $e^{\beta\Phi_r}$, where only terms of the order $\alpha^{N-2k}\beta^k$, $k=0,\dots,\left[\frac{N}{2}\right]$ are non-vanishing inside $\langle N+L|\dots|L\rangle$. The term $e^{\alpha\Phi_r}$ should be considered in a way similar to [5] (see Appendix). Notice the appearance of the integration domain $\mathbb{M}_{2k,N-2k}$ when we re-write the product of pairwise integrals as an N-fold integral. Next step: we send Γ_+ to the right and Γ_- to the left taking into account relations $\langle N+L|\Gamma_-(\mathbf{s}) = \langle N+L|$ and $\Gamma_+(\mathbf{t})|L\rangle = |L\rangle$, also using (23). In this way we get rid of the operators Γ_+ and Γ_-, and obtain factors e^V inside integrals instead, the integrals are still between vacuums of different charges, $\langle N+L|$ and $|L\rangle$. At last, we get rid of fermions thanks to

$$\langle N+L|\psi(z_1)\cdots\psi(z_N)|L\rangle = z_1^L\cdots z_N^L\Delta_N(\mathbf{z}),\quad \Delta_N(\mathbf{z}) := \prod_{i<j\le N}(z_i-z_j) \tag{29}$$

obtained by a simple direct calculation. We obtain (26).

Now we see that the fermionic expectation value $\langle N+L|\,\Gamma_+(\mathbf{t})\,e^{\alpha\Phi_c}e^{\Phi_r}\,\Gamma_-(\mathbf{s})\,|L\rangle$, which we obtained, is an example of a 2-BKP tau function with respect to the variables $N, L, \mathbf{t}$, as introduced in [4] where Hirota equations for such tau functions were written down. By symmetry, it is also a tau function with respect to the variables $N, L, \mathbf{s}$. Such ("coupled" BKP, or, the same 2-BKP) tau functions were considered in [3].

Thus, we establish that the deformed real Ginibre ensemble (7) is the subject to the theory of integrable systems.

3 Quaternion-real Ginibre ensemble and its deformation

Each $N\times N$ quaternionic matrix (that is, a matrix with quaternion entries) may be also viewed as a $2N\times 2N$ matrix with complex number entries if the quaternions e_n, $n=1,2,3,4$, are realized as 2×2 matrices:

$$e_0 = \begin{pmatrix}1&0\\0&1\end{pmatrix},\quad e_1 = \begin{pmatrix}i&0\\0&-i\end{pmatrix},\quad e_2 = \begin{pmatrix}0&1\\-1&0\end{pmatrix},\quad e_3 = \begin{pmatrix}0&i\\i&0\end{pmatrix} \tag{30}$$

A quaternionic matrix is a matrix where each entry is a linear combination of quaternions with real coefficients. Being re-written as $2N\times 2N$ matrix X with complex entries, it satisfies the relation

$$EXE = -\bar X \tag{31}$$

where the bar denotes complex conjugation and where E is the block-diagonal $2N \times 2N$ matrix with matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on the main diagonal; this characterization follows from the explicit expression (30) and from $e_2 e_j e_2 = e_j$, $j = 1, 3$, and also from $e_2 e_j e_2 = -e_j$, $j = 0, 2$. Below we shall treat quaternionic matrices as $2N \times 2N$ matrices with the defining property (31).

Quaternionic Hermitian ensemble. First, we recall the case where the quaternionic matrix X is Hermitian (in this case X is called self-dual). The ensemble of random self-dual matrices is called the symplectic ensemble, the Gaussian symplectic ensemble (G-SE) is called the Dyson-Wigner symplectic ensemble. We will consider a deformation (d-SE) of the symplectic ensemble as follows

$$I_N^{d-SE}(L, \mathbf{t}) = \int d\mu(X, L, \mathbf{t}), \qquad d\mu(X, L, \mathbf{t}) = \det X^L \, e^{-\mathrm{tr} X^2 + 2\mathrm{tr} V(X, \mathbf{t})} \, d\mu(X) \tag{32}$$

where $d\mu(X)$ is the standard measure on the space of self-dual matrices (for details see [11]). As in the orthogonal ensemble case, we may restrict to the subspace of invertible matrices and consider the double deformed restricted ensemble of quaternionic matrices (dd-SE),

$$I_N^{dd-SE}(L, \mathbf{t}, \mathbf{s}, \Pi) = \int d\mu(X, L, \mathbf{t}, \mathbf{s}, \Pi), \qquad d\mu(X, L, \mathbf{t}, \mathbf{s}, \Pi) = \Pi(X) \det X^L e^{-\mathrm{tr} X^2 + 2\mathrm{tr} V(X, \mathbf{t}) - 2\mathrm{tr} V(X^{-1}, \mathbf{s})} \, d\mu(X), \tag{33}$$

where we add complex deformation parameters $\mathbf{s} = (s_2, s_2, \ldots)$ and an integer L. Similar to the case of orthogonal ensembles, we introduce the additional freedom

$$\Pi(X) = \prod_{i=1}^{N} \pi(x_i).$$

Here N pairs $x_1, x_1, x_2, x_2 \ldots, x_N, x_N$ are the eigenvalues of the self-dual random $2N \times 2N$ matrix X. Being re-written as an integral over eigenvalues of Hermitian quaternionic matrices X [11], it is as follows

$$I_N^{dd-SE}(L, \mathbf{t}, \mathbf{s}, \Pi) = d_N \int (\Delta(x_1, x_2, \ldots, x_N))^4 \prod_{i=1}^{N} x_i^{2L} e^{-x_i^2 + 2V(x_i, \mathbf{t}) - 2V(x_i^{-1}, \mathbf{s})} \pi(x_i) dx_i \tag{34}$$

where the factor d_N does not depend on the deformation parameters.

Quaternionic non-Hermitian ensemble or, the same, the so-called **quaternion-real ensemble**. The Gaussian quaternion-real Ginibre ensemble, also known as (Gaussian) Ginibre symplectic ensemble (G-GinSE) is defined on the space of quaternionic matrices by assigning the same Gaussian probability measure to each entry:

$$I_N^{G-GinSE} = \int d\mu(X), \qquad d\mu(X) = e^{-\frac{1}{2}\mathrm{tr}(XX^\dagger)} \prod_{i,j} dX_{ij} \tag{35}$$

where X is treated as $2N \times 2N$ matrix.

The measures (35) and (33) are invariant under symplectic transformations.

The ensemble (35) is a non-Hermitian analogue of the ensemble of random quaternionic self-dual matrices (known as Dyson-Wigner symplectic ensemble), see [11] for details.

The elliptic deformation of G-GinSE measure in [1] is quite similar to (6): $d\mu(X) \to e^{-a\mathrm{tr}\left(X^2 + (X^\dagger)^2\right)} d\mu(X)$, where a is a deformation parameter.

The partition function for the deformed quaternion-real Ginibre ensemble (d-GinSE) will be defined as

$$I_N^{d-GinSE}(L, \mathbf{t}, \Pi) = \int d\mu(X, L, \mathbf{t}, \Pi), \quad d\mu(X, L, \mathbf{t}, \Pi) := \Pi(X) \det X^L \, e^{\mathrm{tr} V(X, \mathbf{t})} \, d\mu(X) \tag{36}$$

where the non-negative integer L and the set of numbers $\mathbf{t} = (t_1, t_2, \ldots)$ are deformation parameters. $\Pi(X)$ is chosen in the same way as in the case of real Ginibre ensemble,

$$\Pi(X) = \prod_{i=1}^{m} \pi(z_i, \bar{z}_i) \prod_{i=2m+1}^{2N} \pi(x_i) \tag{37}$$

Actually, real eigenvalues are of degree no less than two, and do not contribute to the partition function [1].

On the space of invertible quaternionic matrices we consider the double deformed Ginibre ensemble

$$I_N^{dd-GinSE}(L, \mathbf{t}, \mathbf{s}, \Pi) = \int d\mu(X, L, \mathbf{t}, \mathbf{s}, \Pi)\,, \quad d\mu(X, L, \mathbf{t}, \mathbf{s}, \Pi) = \Pi(X) \det X^L \, e^{\mathrm{tr} V(X,\mathbf{t}) - \mathrm{tr} V(X^{-1},\mathbf{s})} \, d\mu(X) \tag{38}$$

where the integer L and and the sets $\mathbf{t} = (t_1, t_2, \ldots)$, $\mathbf{s} = (s_1, s_2, \ldots)$ are deformation parameters. Here we assume that the integral is either convergent or regularized, see Remark 1.

Notice that if one takes

$$t_n \to t_n - \frac{1}{2n} \sum_{i=1} a_i p_i^n, \qquad s_n \to s_n - \frac{1}{2n} \sum_j b_j q_j^n \tag{39}$$

then the integral (36) reads

$$\int \det X^L \prod_i \det\left(1 - p_i X\right)^{a_i} \prod_j \det\left(1 - q_i X^{-1}\right)^{-b_i} \, d\mu(X, L, \mathbf{t}, \mathbf{s}, \Pi) \tag{40}$$

As in the case of real matrices, all complex eigenvalues occur in complex conjugate pairs. Real eigenvalues have multiplicities no less than two. A calculation similar to the calculation for real matrices yields

$$I_N^{dd-GinSE}(L, \mathbf{t}, \mathbf{s}, \Pi) = \tag{41}$$

$$c_N \int_{\mathbb{M}_N} \Delta_{2N}(z_1, \bar{z}_1, z_2, \bar{z}_2, \ldots, z_N, \bar{z}_N) \prod_{i=1}^{N} |z_i - \bar{z}_i||z_i|^{2L} e^{V(z_i,\mathbf{t}) + V(\bar{z}_i,\mathbf{t}) - V(z_i^{-1},\mathbf{s}) - V(\bar{z}_i^{-1},\mathbf{s}) - |z_i|^2} \pi(z_i, \bar{z}_i) d^2 z_i$$

with some constant c_N, where the integration domain $\mathbb{M}_n$ consists of the sets of $\mathbf{z}$ which satisfy $\Re z_i > \Re z_{i+1}$ and $\Im z_i > 0$, with V given by (2), see [1], [29], [26], [11]. Compared to [1], [29], [26], [11] we include the deformation given by the parameters L, $\mathbf{t}$ and $\mathbf{s}$.

Theorem 2. *Introduce*

$$c(\mathbf{t}, \mathbf{s}) J_N(L, \mathbf{t}, \mathbf{s}, \Pi; \alpha, \beta) = \langle 2N + L | \, \Gamma_+(\mathbf{t}) \, e^{\alpha \Phi_c} e^{\beta \Phi_r} \, \Gamma_-(\mathbf{s}) \, |L\rangle \tag{42}$$

where

$$\Phi_c^q = \int_{\mathbb{C}_+} (\bar{z} - z) \psi(z) \psi(\bar{z}) e^{-|z|^2} \pi(z, \bar{z}) d^2 z \tag{43}$$

is the integral over the upper half of complex plane and

$$\Phi_r^q = \int_{\mathbb{R}} \frac{\partial \psi(x)}{\partial x} \psi(x) e^{-x^2} \pi(x) dx \tag{44}$$

Then

$$I_N^{d-GinSE}(L, \mathbf{t}, \Pi) = d_N J_N(L, \mathbf{t}, \mathbf{s} = 0, \Pi; \alpha = 1, \beta = 0) \tag{45}$$

$$I_N^{dd-GinSE}(L, \mathbf{t}, \mathbf{s}, \Pi) = d_N J_N(L, \mathbf{t}, \mathbf{s}, \Pi; \alpha = 1, \beta = 0)\,, \quad I^{dd-SE}(L, \mathbf{t}, \mathbf{s}, \Pi) = c_N J_N(L, \mathbf{t}, \mathbf{s}, \Pi; \alpha = 0, \beta = 1) \tag{46}$$

Therefore $I_N^{d-GinSE}(L, \mathbf{t}, \Pi)$ is an example of the large ("fermionic") BKP tau function of [4] with respect to the variables $N, L, \mathbf{t}$. Thus $c(\mathbf{t}, \mathbf{s}) I_N^{dd-GinSE}(L, \mathbf{t}, \mathbf{s}, \Pi)$ and $c(\mathbf{t}, \mathbf{s}) I_N^{dd-SE}(L, \mathbf{t}, \mathbf{s}, \Pi)$ are examples of the large 2-BKP tau function considered in [3] with respect to the variables $N, L, \mathbf{t}, \mathbf{s}$ and of large 2-DKP tau function introduced in [21], see Appendix A.4.

Let us note that the presentation with $\mathbf{s} = 0$, namely, the representation for the symplectic Dyson-Wigner ensemble (for $\alpha = 0$) was found by J. van de Leur in [5].

4 Gaussian complex Ginibre ensemble and its deformation

The complex Ginibre ensemble is considered to be the easiest and it is relatively the most studied one, and it is quite similar to the well-known model of normal (i.e. diagonalizable) matrices [17]; therefore I will skip details.

Hermitian random matrices. First, we recall that the ensemble of random Hermitian matrices X with gaussian distribution for each entry, is known as the Gaussian unitary ensemble (GUE). The deformation of the measure, resulting from multiplication by the Gaussian factor $e^{-\mathrm{tr}X}$ by $e^{\mathrm{tr}V(X,\mathbf{t})}$, was related to integrable systems in [13]. Let us include additional deformation parameters $L, \mathbf{s}$. Then the partition function of the double deformed ensemble of Hermitian matrices (or, the same, of double deformed unitary ensemble) is

$$I_N^{dd-UE}(L,\mathbf{t},\mathbf{t}',\mathbf{s},\mathbf{s}') = \int \det(X)^L e^{\sum_{m=1}\left((t_m+t'_m)\mathrm{tr}X^m-(s_m+s'_m)\mathrm{tr}X^{-m}\right)-\mathrm{tr}X^2} d\mu(X) \tag{47}$$

where $d\mu(X)$ is the Haar measure on the space of Hermitian matrices (see [11] for details). Actually we do not need $\mathbf{t}'$ and $\mathbf{s}'$ but we keep them so as to compare the model with the non-Hermitian one. The fermionic representation for this model (with $\mathbf{s} = \mathbf{s}' = 0$) was given in [14] (see also [16]). However to compare the result with the non-Hermitian case, it is suitable to use another fermionic construction suggested in [15],

$$I_N^{dd-UE}(L,\mathbf{t},\mathbf{t}',\mathbf{s},\mathbf{s}') = e_N\langle L+N,-N|\,\Gamma(\mathbf{t},\mathbf{t}')\,e^{\int_{\mathbb{C}}\psi^{(1)}(x)\psi^{\dagger(2)}(x)e^{-x^2}dx}\,\bar{\Gamma}(\mathbf{s},\mathbf{s}')\,|L,0\rangle \tag{48}$$

with an extra factor e_N, independent of the deformation parameters. Compared to [15] we include additional dependence on $L, \mathbf{s}$.

Non-Hermitian random matrices The partition function for the complex restricted Ginibre ensemble with the double deformed measure (or, the same, of the double deformed Ginibre unitary ensemble)

$$I_N^{dd-GinUE}(L_1,L_2,\mathbf{t},\mathbf{t}',\mathbf{s},\mathbf{s}') = \tag{49}$$

$$\int \det(X)^{L_1}\det(X^\dagger)^{-L_2} e^{\sum_{m=1}\left(t_m\mathrm{tr}X^m+t'_m\mathrm{tr}(X^\dagger)^m-s_m\mathrm{tr}X^{-m}-s'_m\mathrm{tr}(X^\dagger)^{-m}\right)-\mathrm{tr}XX^\dagger}\prod_{i,j}\Pi(X)dX_{ij}$$

is equal to the following two-component 2D Toda lattice [9] tau function

$$I_N^{dd-GinUE}(L_1,L_2,\mathbf{t},\mathbf{t}',\mathbf{s},\mathbf{s}') = f_N\langle L_1+N,L_2-N|\,\Gamma(\mathbf{t},\mathbf{t}')\,e^{\int_{\mathbb{C}}\psi^{(1)}(z)\psi^{\dagger(2)}(\bar{z})e^{-|z|^2}\pi(z,\bar{z})d^2z}\,\bar{\Gamma}(\mathbf{s},\mathbf{s}')\,|L_1,L_2\rangle \tag{50}$$

where the parameters $L_1, L_2, \mathbf{t}, \mathbf{t}', \mathbf{s}, \mathbf{s}'$ are all chosen in such a way that the integral is either convergent or may be regularized (see Remark 1). The factor f_N is independent of $L_1, L_2, \mathbf{t}, \mathbf{t}', \mathbf{s}, \mathbf{s}'$.

This tau function coincides with the tau function related to the partition function of the model of normal matrices previously used in [18] in the context of two-matrix and normal matrix models. According to the formula (9.7) of the seminal paper [9]

Remark 3. The representation (50) results in the Toda lattice equations

$$\frac{1}{2}D_{t_1}D_{s_1}I_N^{dd-GinUE}(L_1,L_2)\cdot I_N^{dd-GinUE}(L_1,L_2) =$$

$$\left(I_N^{dd-GinUE}(L_1,L_2)\right)^2 - I_N^{dd-GinUE}(L_1+1,L_2)I_N^{dd-GinUE}(L_1-1,L_2)$$

and

$$\frac{1}{2}D_{t'_1}D_{s'_1}I_N^{dd-GinUE}(L_1,L_2)\cdot I_N^{dd-GinUE}(L_1,L_2) =$$

$$\left(I_N^{dd-GinUE}(L_1,L_2)\right)^2 - I_N^{dd-GinUE}(L_1,L_2+1)I_N^{dd-GinUE}(L_1,L_2-1)$$

where D_x denotes the Hirota derivative with respect to a variable x. These relations for the two-component and two-sided tau function in the right-hand side of (50) were written down in the seminal paper [9] (see the formula (9.7) there).

Perturbation series in deformation parameters for the complex Ginibre ensemble are basically the same as found in [18].

4.1 On perturbation series in deformation parameters

The deviation of the models considered here from Gaussian ones is described by perturbation series in deformation parameters. The fermionic representation allows this to be done in a straightforward way. For the complex case, this problem was solved in [18]: four sets of deformation parameters $\mathbf{t},\mathbf{t}',\mathbf{t},\mathbf{t}'$ produce a series in a product of four Schur functions over four partitions.

In [3] we considered the following series over partitions

$$\sum_{\substack{\lambda\in\mathrm{P}\\ \ell(\lambda)\le N}} \bar{A}_{h(\lambda)}(L)\, s_\lambda(\mathbf{t}) \tag{51}$$

where s_λ is the Schur function (see Appendix A.3) and $\bar{A}_{h(\lambda)}(L)$ is a certain Pfaffian, see Appendix A.5. Such series one obtains for many kinds of matrix integrals, see [37], [38], for instance, for integrals over symplectic and orthogonal groups, see Appendix A.7. The notations are as follows. $\lambda=(\lambda_1,\dots,\lambda_N)$, $\lambda_1\ge\dots\lambda_N\ge 0$, is a partition, see [12] and $h(\lambda)=\lambda_i-i+N$ are called the shifted parts of λ. The factors $\bar{A}_h$ on the right-hand side of (51) are determined in terms of a pair $(A,a)=:\bar{A}$, where A is an infinite skew symmetric matrix and a is an infinite vector. For a set $h=(h_1,\dots,h_N)$, $h_1>\cdots>h_N$, the numbers $\bar{A}_h$ are defined as the Pfaffian of an antisymmetric $2n\times 2n$ matrix $\tilde{A}$:

$$\bar{A}_h(L) := \mathrm{Pf}[\tilde{A}] \tag{52}$$

where, for $N=2n$ even,

$$\tilde{A}_{ij}=-\tilde{A}_{ji}:=A_{h_i+L,h_j+L},\quad 1\le i<j\le 2n \tag{53}$$

and, for $N=2n-1$ odd,

$$\tilde{A}_{ij}=-\tilde{A}_{ji}:=\begin{cases} A_{h_i+L,h_j+L} & \text{if}\quad 1\le i<j\le 2n-1\\ a_{h_i+L} & \text{if}\quad 1\le i<j=2n\end{cases}. \tag{54}$$

In addition, we set $\bar{A}_0=1$.

Having a fermionic representation, it is straightforward to write down the perturbation series of partition functions of Ginibre ensembles in terms of the Schur functions, as was done in other cases (for instance see [24], [19], [18]). In the context of integrals over matrices, in view of the Cauchy-Littlewood identity, $e^{V(X,\mathbf{t})}=\sum_\lambda s_\lambda(X)s_\lambda(\mathbf{t})$, series over partitions (51) have an additional meaning as $\sum_\lambda\langle s_\lambda(X)\rangle_X|_{\mathbf{t}=0}s_\lambda(\mathbf{t})$, where $\langle\,\rangle_X|_{\mathbf{t}=0}$ means the averaging over a given matrix ensemble evaluated at $\mathbf{t}=0$.

(I) First, let us do it for the quaternion-real Ginibre ensemble. Re-writing

$$\int_{\mathbb{C}_+}(\bar{z}-z)\psi(z)\psi(\bar{z})e^{-|z|^2}e^{-V(z^{-1},\mathbf{s})-V(\bar{z}^{-1},\mathbf{s})}\pi(z,\bar{z})d^2z=\sum_{n,m}A_{nm}\psi_n\psi_m \tag{55}$$

where A_{nm} is the moment matrix

$$A_{nm}^{GinSE}(\mathbf{s},\Pi)=\int_{\mathbb{C}_+} z^n\bar{z}^m(\bar{z}-z)e^{-|z|^2-V(z^{-1},\mathbf{s})-V(\bar{z}^{-1},\mathbf{s})}\pi(z,\bar{z})d^2z,\qquad a_n^{GinSE}=0,\quad n,m=1,\dots,N \tag{56}$$

we reduce the problem to the one solved in [3]. We obtain the following series in the Schur functions (see [12] for definitions and details) s_λ (please, do not confuse s_λ and the deformation parameters s_i):

$$I_N^{dd-GinSE}(L,\mathbf{t},\mathbf{s},\Pi) = \sum_{\substack{\lambda\\ \ell(\lambda)\le 2N}} \bar{A}_{\{h\}}^{GinSE}(L,\mathbf{s},\Pi) s_\lambda(\mathbf{t}) \tag{57}$$

where $h_i = \lambda_i - i + 2N$. In the case $\mathbf{s} = 0$ and $\pi = 1$, we get $A_{m,n} = -A_{n,m} = \frac{1}{2}n!\delta_{m+1,n}$. The Pfaffian $A_{\{h\}}$ of such Jacobian matrices vanishes for all sets $\{h\}$ except the sets related to partitions of form $\lambda = \mu \cup \mu$, where μ is any partition. We call such partition, $\lambda = (\mu_1, \mu_1, \dots, \mu_N, \mu_N,)$, fat partition because each part of λ occurs twice. We obtain

$$I_N^{dd-GinSE}(L,\mathbf{t},\mathbf{s}=0) = \sum_\mu \langle s_{\mu\cup\mu}(X)\rangle_X s_{\mu\cup\mu}(\mathbf{t}) = \sum_{\substack{\mu\\ \ell(\mu)\le N}} 2^{-|\mu|}(2N)_{\mu,4} s_{\mu\cup\mu}(\mathbf{t}) \tag{58}$$

where, for a given partition $\mu = (\mu_1, \dots, \mu_N)$, we define the generalized Pochhammer symbol (we use the same notations as in [3]) :

$$(a)_{\mu,\beta} := (a)_{\mu_1}(a - \frac{1}{2}\beta)_{\mu_2} \cdots (a - \frac{1}{2}N\beta)_{\mu_N}, \quad (a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} \tag{59}$$

Let us note that the relation $\langle s_{\mu\cup\mu}(X)\rangle_X = 2^{-|\mu|}(2N)_{\mu,4}$ was earlier obtained by other means in [31]. A similar relation for the GinOE was obtained in [32].

(II) For the DW symplectic ensemble the formula is the same, but now instead of (56) we have

$$A_{nm}^{SE}(\mathbf{s}) = \frac{n-m}{2}\int_{\mathbb{R}} x^{n+m-1} e^{-x^2 - 2V(x^{-1},\mathbf{s})} dx, \qquad a_n^{SE} = 0, \quad n,m = 1,\dots,N \tag{60}$$

(III) For the DW orthogonal ensemble, in case N is even, the formula is the same, however, for N odd, we need to define $\bar{A}_{\{h\}}$ via the pair A, a. Now

$$A_{nm}^{OE}(\mathbf{s}) = \int_{\mathbb{R}}\int_{\mathbb{R}} x^n y^m \mathrm{sgn}(x-y) e^{-\frac{1}{2}x^2 - \frac{1}{2}y^2 - V(x^{-1},\mathbf{s}) - V(y^{-1},\mathbf{s})} dxdy, \tag{61}$$

$$a_n^{OE}(\mathbf{s}) = \int_{\mathbb{R}} x^n e^{-V(x^{-1},\mathbf{s})} dx \tag{62}$$

(IV) For the real Ginibre ensemble, we have

$$A_{nm}^{GinOE}(\mathbf{s}) = \int_{\mathbb{C}_+} z^n \bar{z}^m \mathrm{erfc}\left(\frac{|z-\bar{z}|}{\sqrt{2}}\right) e^{-\Re z^2 - V(z^{-1},\mathbf{s}) - V(\bar{z}^{-1},\mathbf{s})} d^2 z \tag{63}$$

$$+ \int_{\mathbb{R}}\int_{\mathbb{R}} x^n y^m \mathrm{sgn}(x-y) e^{-\frac{1}{2}x^2 - \frac{1}{2}y^2 - V(x^{-1},\mathbf{s}) - V(y^{-1},\mathbf{s})} dxdy,$$

$$a_n^{GinOE}(\mathbf{s}) = \int_{\mathbb{R}} x^n e^{-V(x^{-1},\mathbf{s})} dx \tag{64}$$

Let us also note that in all cases where $\mathbf{s} = 0$ (namely, for d-OE, d-GinOE, d-SE, d-GinSE ensembles), moments can be explicitly evaluated in a straightforward way.

5 On Pfaffian formulae

The other advantage of the fermionic approach is that it gives a straightforward way to evaluate correlation functions. It is nothing but an application of Wick's rule which yields answers in the form of Pfaffians and determinants. Here some examples will be written down.

Various Pfaffian formulae are known in the study of non-Hermitian matrices, see [28], [1]. These formulae, and perhaps some new ones, may be obtained by applying the Wick rule to the evaluation of the fermionic expectation values, see Appendix A.5.

Example. Let us shift $\mathbf{t}$ as in (11) where all p_i are different, all $b_i = 0$, all $a_i = -1$ and $N_1 = N$ is even. Then according to relation (74) and the Wick rule, one can write the expectation value as a Pfaffian of the pairwise expectation values, see Appendix A.5. Then (12) is

$$\int \prod_i^N \det\left(1 - p_i X\right)^{-1} \, d\mu(X, L, \mathbf{t}, \mathbf{s}) = \frac{\prod_{i=1}^N p_i^{(L+1)(2-N)}}{\prod_{i>j}(p_i - p_j)} \, \mathrm{Pf}\left[K\right]_{n,m=1,\dots,N} \tag{65}$$

where the fermionic representation for the tau function $\tau_N(L, \mathbf{t}, \mathbf{s})$ (25) yields

$$K_{nm} = \langle L|\psi^\dagger(p_n)\psi^\dagger(p_m)e^{\Phi}|L\rangle = (p_m - p_n)K^*_{nm} \tag{66}$$

where K^*_{nm}:

$$K^*_{nm} = \int_{\mathbb{R}} x^L y^L \frac{|x-y| e^{-\frac{1}{2}x^2 - \frac{1}{2}y^2 + V(x,\mathbf{t}) + V(y,\mathbf{t}) - V(x^{-1},\mathbf{s}) - V(y^{-1},\mathbf{s})} dxdy}{(1 - xp_n)(1 - yp_n)(1 - xp_m)(1 - yp_m)} \qquad \text{for OE} \tag{67}$$

$$K^*_{nm} = \int_{\mathbb{C}_+} (z - \bar{z})|z|^{2L} \frac{\mathrm{erfc}\left(\frac{|z-\bar{z}|}{\sqrt{2}}\right) e^{-\Re z^2 + V(z,\mathbf{t}) + V(\bar{z},\mathbf{t}) - V(z^{-1},\mathbf{s}) - V(\bar{z}^{-1},\mathbf{s})} d^2 z}{(1 - zp_n)(1 - \bar{z}p_n)(1 - zp_m)(1 - \bar{z}p_m)} \tag{68}$$

$$+ \int_{\mathbb{R}} x^L y^L \frac{|x-y| e^{-\frac{1}{2}x^2 - \frac{1}{2}y^2 + V(x,\mathbf{t}) + V(y,\mathbf{t}) - V(x^{-1},\mathbf{s}) - V(y^{-1},\mathbf{s})} dxdy}{(1 - xp_n)(1 - yp_n)(1 - xp_m)(1 - yp_m)} \qquad \text{for GinOE} \tag{69}$$

$$K^*_{nm} = \int_{\mathbb{R}} x^{2L} \frac{e^{-x^2 + 2V(x,\mathbf{t}) - 2V(x^{-1},\mathbf{s})} dx}{(1 - xp_n)^2(1 - xp_m)^2} \qquad \text{for SE} \tag{70}$$

$$K^*_{nm} = \int_{\mathbb{C}_+} |z|^{2L} \frac{|z - \bar{z}|^2 e^{-|z|^2 + V(z,\mathbf{t}) + V(\bar{z},\mathbf{t}) - V(z^{-1},\mathbf{s}) - V(\bar{z}^{-1},\mathbf{s})} d^2 z}{(1 - zp_n)(1 - \bar{z}p_n)(1 - zp_m)(1 - \bar{z}p_m)} \qquad \text{for GinSE} \tag{71}$$

Here $\bar{z}$ is the complex conjugate of z. In this example, the number of parameters p_i is equal to N. In other cases, we typically obtain Pfaffians of block matrices. This will be written down in a more detailed text.

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A Appendices

A.1 Vertex operators

Vertex operators we need are as follows

$$\hat{X}(L, \mathbf{t}, \lambda) := e^{\sum_{n=1}^\infty \lambda^n t_n} \lambda^L e^{-\sum_{n=1}^\infty \frac{1}{n\lambda^n}\frac{\partial}{\partial t_n}}, \quad \hat{X}^\dagger(L, \mathbf{t}, \lambda) := e^{-\sum_{n=1}^\infty \lambda^n t_n} \lambda^{-L} e^{\sum_{n=1}^\infty \frac{1}{n\lambda^n}\frac{\partial}{\partial t_n}} \tag{72}$$

$$\hat{Y}(L, \mathbf{s}, \lambda) := e^{-\sum_{n=1}^\infty \lambda^{-n} s_n} \lambda^L e^{\sum_{n=1}^\infty \frac{\lambda^n}{n}\frac{\partial}{\partial s_n}}, \quad \hat{Y}^\dagger(L, \mathbf{s}, \lambda) := e^{\sum_{n=1}^\infty \lambda^{-n} s_n} \lambda^{-L} e^{-\sum_{n=1}^\infty \frac{\lambda^n}{n}\frac{\partial}{\partial s_n}} \tag{73}$$

The formula which relates fermions to bosons first was found in [23].

The following bosonization relation is useful

$$\langle L+N|\Gamma_+(\mathbf{t}+\sum_{i=1}^{N}[p_i]) = \frac{\langle L|\psi^\dagger(p_1^{-1})\cdots\psi^\dagger(p_N^{-1})\Gamma_+(\mathbf{t})}{\prod_{i=1}^{N}p_i^{(L+1)(N-1)}\prod_{i>j}(p_i-p_j)} \tag{74}$$

Introduce

$$\hat{\Omega}_n(L,\mathbf{t}) = \operatorname*{res}_{\lambda}\left(\frac{\partial^n\hat{X}(L,\mathbf{t},\lambda)}{\partial\lambda^n}\hat{X}^\dagger(L,\mathbf{t},\lambda)\right), \quad \hat{\Omega}_n^*(L,\mathbf{s}) = \operatorname*{res}_{\lambda}\left(\hat{Y}^\dagger(L,\mathbf{s},\lambda)\frac{\partial^n\hat{Y}(L,\mathbf{s},\lambda)}{\partial\lambda^n}\right) \tag{75}$$

A.2 Fermions

We shall recall some facts and notations of [9]. Introduce free fermionic fields $\psi(z)=\sum_{i\in\mathbb{Z}}\psi_i z^i$, $\psi^\dagger(z)=\sum_{i\in\mathbb{Z}}\psi^\dagger_{-i-1}z^i$ whose Fourier components anti-commute as follows $\psi_i\psi_j+\psi_j\psi_i=\psi_i^\dagger\psi_j^\dagger+\psi_j^\dagger\psi_i^\dagger=0$ and $\psi_i\psi_j^\dagger+\psi_j^\dagger\psi_i=\delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker symbol. We put

$$\psi_i|0\rangle = \psi^\dagger_{-i}|0\rangle = \langle 0|\psi_{-i} = \langle 0|\psi_i^\dagger = 0 \tag{76}$$

where $\langle 0|$ and $|0\rangle$ are left and right vacuum vectors of the fermionic Fock space, $\langle 0|\cdot 1\cdot|0\rangle = 1$. Also introduce

$$\langle n| = \begin{cases}\langle 0|\psi_0^\dagger\cdots\psi_{n-1}^\dagger & \text{if} \quad n>0\\ \langle 0|\psi_{-1}\cdots\psi_{-n} & \text{if} \quad n<0\end{cases}, \quad |n\rangle = \begin{cases}\psi_{n-1}\cdots\psi_0|0\rangle & \text{if} \quad n>0\\ \psi^\dagger_{-n}\cdots\psi^\dagger_{-1}|0\rangle & \text{if} \quad n<0\end{cases} \tag{77}$$

Then $\langle n|\cdot 1\cdot|m\rangle = \delta_{n,m}$.

Following [4], we introduce an additional Fermi mode, which we shall denote by ϕ, with properties [5]

$$\phi\psi_i+\psi_i\phi = \phi\psi_i^\dagger+\psi_i^\dagger\phi = 0, \quad \phi^2=\frac{1}{2} \tag{78}$$

$$\phi|0\rangle = |0\rangle\frac{1}{\sqrt{2}}, \qquad \langle 0|\phi = \frac{1}{\sqrt{2}}\langle 0| \tag{79}$$

such that $\langle L|\phi|L\rangle = \frac{(-)^L}{\sqrt{2}}$.

Now, two-component Fermi fields used in Section 4 are defined as

$$\psi^{(i)}(z) = \sum_{n\in\mathbb{Z}} z^n\psi_{2n+i}, \quad \psi^{(i)\dagger}(z) = \sum_{n\in\mathbb{Z}} z^{-n-1}\psi^\dagger_{2n+1} \tag{80}$$

Other details about the two-component fermions may be found in [9], or in [18].

We have

$$\hat{X}(L,\mathbf{t},\lambda)X^\dagger(L,\mu)\langle N+L|\Gamma_+(\mathbf{t})g\Gamma_-(\mathbf{s})|L\rangle = \langle N+L|\Gamma_+(\mathbf{t})\psi(\lambda)\psi^\dagger(\mu)g\Gamma_-(\mathbf{s})|L\rangle \tag{81}$$

$$\hat{Y}^\dagger(L,\mathbf{s},\mu)Y(\lambda)\langle N+L|\Gamma_+(\mathbf{t})g\Gamma_-(\mathbf{s})|L\rangle = \langle N+L|\Gamma_+(\mathbf{t})g\psi(\lambda)\psi^\dagger(\mu)\Gamma_-(\mathbf{s})|L\rangle \tag{82}$$

Then, it follows that

$$\hat{\Omega}_n(L,\mathbf{t})\langle N+L|\Gamma_+(\mathbf{t})g\Gamma_-(\mathbf{s})|L\rangle = \langle N+L|\Gamma_+(\mathbf{t})\tilde{\Omega}_n g\Gamma_-(\mathbf{s})|L\rangle \tag{83}$$

$$\hat{\Omega}_n(L,\mathbf{t})\langle N+L|\Gamma_+(\mathbf{t})g\Gamma_-(\mathbf{s})|L\rangle = \langle N+L|\Gamma_+(\mathbf{t})g\tilde{\Omega}_n\Gamma_-(\mathbf{s})|L\rangle \tag{84}$$

[5] In notations of [4] our ψ_n, $\psi_n^\dagger$ and ϕ read respectively $\psi_{n+\frac{1}{2}}$, $\psi^\dagger_{n+\frac{1}{2}}$ and ψ_0

where

$$\tilde{\Omega}_n = \operatorname*{res}_{\lambda} \left(\frac{\partial^n \psi(\lambda)}{\partial \lambda^n} \psi^\dagger(\lambda) \right) \tag{85}$$

Using the fermionic representation, one may verify that tau functions, related to the considered ensembles (dd-OE, dd-GinOE, dd-SE, dd-GinSE), obey the constraints

$$\left(\hat{\Omega}_n(L,\mathbf{t}) - \hat{\Omega}^*_n(L,\mathbf{s})\right) \tau(L,\mathbf{t}',\mathbf{s}') = 0, \quad n \geq 1 \quad \text{odd} \tag{86}$$

where $t'_k = t_k - \frac{1}{2}\delta_{2,k}$ $s'_k = s_k - \frac{1}{2}\delta_{2,k}$ (this shift appears due to the Gaussian measure in the undeformed ensembles).

A.3 The Schur function

Consider polynomials $h_n(\mathbf{t})$ defined by $e^{\sum_{n=1}^\infty z^n t_n} = \sum_{n=0}^\infty z^n h_n(\mathbf{t})$. Then, the Schur function labeled by a partition $\lambda = (\lambda_1, \dots, \lambda_k > 0)$ may be defined as $s_\lambda(\mathbf{t}) = \det\left(h_{\lambda_i - i + j}(\mathbf{t})\right)_{i,j=1,\dots,k}$. The notation $s_\lambda(X)$ denotes (A.3) where $\mathbf{t} = \sum_i [x_i]$, where x_i are the eigenvalues of X.

A.4 From 2-BKP (25) to TL, BKP, 2-DKP, DKP

Let us consider the large 2-BKP tau function (25).

The general expression for Φ in (25) is

$$\Phi = \sum_{i,j\in\mathbb{Z}} A_{ij}\psi_i\psi_j + \sum_{i,j\in\mathbb{Z}} B_{ij}\psi^\dagger_i\psi^\dagger_j + \sum_{i,j\in\mathbb{Z}} D_{ij}\psi_i\psi^\dagger_j + \phi\sum_{i\in\mathbb{Z}} a_i\psi_i + \phi\sum_{i\in\mathbb{Z}} b_i\psi^\dagger_i \tag{87}$$

To get TL tau function [9], [20] we put all A_{ij}, B_{ij}, a_i, b_i and N to be zero in (25).

To get BKP tau function [4] we put $\mathbf{s} = 0$.

To get 2-DKP tau function [21] we put all $a_i = b_i = 0$.

To get DKP tau function [9] we put $\mathbf{s} = 0$ and all $a_i = b_i = 0$.

Tau functions (26) and (42) correspond to the case where all B_{ij}, D_{ij}, b_i vanish (and for (42) also all $a_i = 0$). The case where only A_{ij} (and perhaps a_i) are non-vanishing is characterized by the condition that the so-called wave functions,

$$w^{(\infty)}(N,L,\mathbf{t},\mathbf{s},z) = \frac{\hat{X}(L,\mathbf{t},z)\tau_N(L,\mathbf{t},\mathbf{s})}{\tau_N(L,\mathbf{t},\mathbf{s})} = \lambda^L e^{V(z,\mathbf{t})-V(z^{-1},\mathbf{s})} P_N(L,\mathbf{t},\mathbf{s},,z) \tag{88}$$

$$w^{*(0)}(N,L,\mathbf{t},\mathbf{s},\lambda) = \frac{\hat{Y}^\dagger(L,\mathbf{s},z)\tau_N(L,\mathbf{t},\mathbf{s})}{\tau_N(L,\mathbf{t},\mathbf{s})} = \lambda^L e^{V(z,\mathbf{t})-V(z^{-1},\mathbf{s})} Q_N(L,\mathbf{t},\mathbf{s},z), \tag{89}$$

are equal, and $P_N = Q_N$ are polynomials in λ of the order N.

If tau functions may be written in form (25) where

$$\Phi = \int \psi(z)\psi(z')d\mu(z,z') \tag{90}$$

then, from the fermionic representation it follows, that the so-called adjoint wave functions,

$$w^{*(\infty)}(N,L,\mathbf{t},\mathbf{s},z) = \frac{\hat{X}^\dagger(L,\mathbf{t},z)\tau_N(L,\mathbf{t},\mathbf{s})}{\tau_N(L,\mathbf{t},\mathbf{s})} \tag{91}$$

$$w^{(0)}(N,L,\mathbf{t},\mathbf{s},z) = \frac{\hat{Y}(L,\mathbf{s},z)\tau_N(L,\mathbf{t},\mathbf{s})}{\tau_N(L,\mathbf{t},\mathbf{s})}, \tag{92}$$

may be related to the wave function $w^{(\infty)}$ by

$$w^{*(\infty)}(N,L,\mathbf{t},\mathbf{s},z) - w^{(0)}(N,L,\mathbf{t},\mathbf{s},z) = N\int \frac{w^{(\infty)}(N-1,L,\mathbf{t},\mathbf{s},k)e^{V(z_1,\mathbf{t})+V(k,\mathbf{t})-V(z_1^{-1},\mathbf{s})-V(k_1^{-1},\mathbf{s})}d\mu(z',k)}{z'-z} \tag{93}$$

A.5 Pfaffians

If A is an anti-symmetric matrix of an odd order, its determinant vanishes. For even order, say k, the following multilinear form in $A_{ij}, i < j \le k$,

$$\mathrm{Pf}[A] := \sum_{\sigma} \mathrm{sgn}(\sigma)\, A_{\sigma(1),\sigma(2)} A_{\sigma(3),\sigma(4)} \cdots A_{\sigma(k-1),\sigma(k)}, \tag{94}$$

where the sum runs over all permutation restricted by

$$\sigma :\, \sigma(2i-1) < \sigma(2i), \quad \sigma(1) < \sigma(3) < \cdots < \sigma(k-1), \tag{95}$$

coincides with the square root of $\det A$, and is called the *Pfaffian* of A, see, for instance, [11]. As one can see, the Pfaffian contains $1 \cdot 3 \cdot 5 \cdot \cdots \cdot (k-1) =: (k-1)!!$ terms.

Wick's relations. Let each of w_i be a linear combination of Fermi operators:

$$\hat{w}_i = \sum_{m\in\mathbb{Z}} v_{im}\psi_m + \sum_{m\in\mathbb{Z}} u_{im}\psi^\dagger_m\,, \quad i = 1,\ldots,n$$

Then the Wick formula is

$$\langle l|\hat{w}_1 \cdots \hat{w}_n|l\rangle = \begin{cases} \mathrm{Pf}\,[A]_{i,j=1,\ldots,n} & \text{if n is even} \\ 0 & \text{otherwise} \end{cases} \tag{96}$$

where A is $n \times n$ antisymmetric matrix with entries $A_{ij} = \langle l|\hat{w}_i\hat{w}_j|l\rangle\,, \quad i < j$.

A.6 Hirota equations

Hirota equations for the large BKP hierarchy were written in [4]. For 2-BKP hierarchy Hirota equations are as follows [3]

$$\begin{aligned} &\oint \frac{dz}{2\pi i} z^{N'+l'-N-l-2} e^{V(\mathbf{t}'-\mathbf{t},z)} \tau_{N'-1}(l',\mathbf{t}'-[z^{-1}],\mathbf{s}')\tau_{N+1}(l,\mathbf{t}+[z^{-1}],\mathbf{s}) \\ &+ \oint \frac{dz}{2\pi i} z^{N+l-N'-l'-2} e^{V(\mathbf{t}-\mathbf{t}',z)} \tau_{N'+1}(l',\mathbf{t}'+[z^{-1}],\mathbf{s}')\tau_{N-1}(l,\mathbf{t}-[z^{-1}],\mathbf{s}) \\ &= \oint \frac{dz}{2\pi i} z^{l'-l} e^{V(\mathbf{s}'-\mathbf{s},z^{-1})} \tau_{N'-1}(l'+1,\mathbf{t}',\mathbf{s}'-[z])\tau_{N+1}(l-1,\mathbf{t},\mathbf{s}-[z]) \\ &+ \int \frac{dz}{2\pi i} z^{l-l'} e^{V(\mathbf{s}'-\mathbf{s},z^{-1})} \tau_{N'+1}(l'-1,\mathbf{t}',\mathbf{s}'+[z])\tau_{N-1}(l+1,\mathbf{t},\mathbf{s}+[z]) \\ &+ \frac{(-1)^{l'+l}}{2}(1-(-1)^{N'+N})\tau_{N'}(l',\mathbf{t}',\mathbf{s}')\tau_N(l,\mathbf{t},\mathbf{s}) \end{aligned} \tag{97}$$

The difference Hirota equation may be obtained from the previous one [3]

$$\begin{aligned} &-\frac{\beta}{\alpha-\beta}\tau_N(l,\mathbf{t}+[\beta^{-1}])\tau_{N+1}(l,\mathbf{t}+[\alpha^{-1}]) - \frac{\alpha}{\beta-\alpha}\tau_N(l,\mathbf{t}+[\alpha^{-1}])\tau_{N+1}(l,\mathbf{t}+[\beta^{-1}]) \\ &+ \frac{1}{\alpha\beta}\tau_{N+2}(l,\mathbf{t}+[\alpha^{-1}]+[\beta^{-1}])\tau_{N-1}(l,\mathbf{t}) = \tau_{N+1}(l,\mathbf{t}+[\alpha^{-1}]+[\beta^{-1}])\tau_N(l,\mathbf{t}). \end{aligned} \tag{98}$$

A.7 Integrals over orthogonal and symplectic groups

Using

$$\int_{O\in\mathbb{O}(N)} s_\lambda(O) d_*O = \begin{cases} 1 & \lambda \text{ is even} \\ 0 & \text{otherwise} \end{cases}, \qquad \int_{S\in\mathbb{S}p(N)} s_\lambda(S) d_*S = \begin{cases} 1 & \lambda^{tr} \text{ is even} \\ 0 & \text{otherwise} \end{cases} \tag{99}$$

where λ^{tr} is the partition conjugate of λ, see [12], we get

$$J_1(\mathbf{t},N) := \int_{O\in\mathbb{O}(N)} e^{\sum_{m=1}^\infty t_m \mathrm{Tr} O^m} d_*O = \sum_{\substack{\lambda \text{ even} \\ \ell(\lambda)\le N}} s_\lambda(\mathbf{t}) \tag{100}$$

$$J_2(\mathbf{t},N) := \int_{S\in\mathbb{S}p(2n)} e^{\sum_{m=1}^\infty t_m \mathrm{Tr} S^m} d_*S = \sum_{\substack{\lambda^{tr} \text{ even} \\ \ell(\lambda)\le 2n}} s_\lambda(\mathbf{t}) \tag{101}$$

The right-hand sides were obtained in [3] as examples of the BKP tau function. Thus, integrals $J_1(\mathbf{t},N)$ and $J_2(\mathbf{t},N)$ are tau functions, N and $\mathbf{t}$ being the BKP higher times.

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