# Local Statistics of Lyapunov Exponents: From GUE to picket fence


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joint work with Zdzisław Burda & Mario Kieburg [J. Phys. A **47** (2014) & arXiv: 1809:05905]

# Outline

- Lyapunov exponents from products of $M$ random matrices of size $N \times N$
- Double scaling limit $M, N \to \infty$:
  Transition between GUE and picket fence statistics
- Summary and open questions

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- [Furstenberg, Kesten 60]: choose $X_{j=1,\dots,M}$ independent $N \times N$ **Gaussian random matrices** (or [Janik, Wieczorek 04; Narayanan, Neuberger 07, Blaizot, Nowak 08; Gudowska-Nowak et al. 03; …])

Can we determine the spectral statistics of $Y^\dagger Y$ (or $L$)?

# Spectral statistics of Lyapunov exponents $\mu_j$

- e.g. density of $\mu_j$ and correlations amongst them
- What kind of questions can we ask?
  - distinguish **global vs. local statistics** (for $N \to \infty$)
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- (When) will we find the same as for $M = 1$ random matrix, i.e. **universality?**

# Tools: Determinantal point process

- simplest choice: $M$ **complex Ginibre matrices**

$$\boxed{\mathcal{P}(X_j) \sim \exp\left[-\mathrm{Tr}\left(X_j^\dagger X_j\right)\right]}, \forall j = 1, \ldots, M$$

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- joint density of singular values[2] $s_a$ of $Y$ form **determinental point process** [GA, Kieburg, Wei 13]

$$\boxed{\mathcal{P}_N(\{s\}) \sim \ \Delta_N(\{s\}) \det_{1\leq b,c\leq N}\left[G^{M,0}_{0,M}\left(\left.\begin{matrix}-\\0,\ldots,0,b-1\end{matrix}\right| s_c\right)\right]}$$

with Vandermonde determinant $\Delta_N(\{s\}) = \det\left[s_a^{b-1}\right]_{a,b=1}^N$

and Mejier $G$-function $G^{M,0}_{0,M}$

- example for biorthogonal ensemble [Borodin 98]

# Kernel and correlation functions

- for determinental point process with kernel $K_N(x, y)$:

$\Rightarrow$ all $k$**-point correlation functions** known

$$R_k(s_1, \ldots, s_k) \equiv \frac{N!}{(N-k)!} \int ds_{k+1} \cdots \int ds_N \mathcal{P}_N(\{s\}) = \det[K_N(s_b, s_c)]_{b,c=1}^{k}$$

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\end{aligned}
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e.g. density $R_1(s) = K_N(s, s)$

- kernel $\boxed{K_N(x, y) = \sum_{j=0}^{N-1} P_j(x) G_j(y)}$ for product of $M$ complex Ginibre matrices with $M, N$ fixed [GA, Kieburg, Ipsen 13]

- known for different products e.g. truncated unitary [Kieburg, Kuijlaars, Stivigny 15]

# Limit ii): Know results for $M \geq 1$ fixed & limit $N \to \infty$

- **global spectrum:**
  resolvent $G(z) = \int \frac{\rho(x)dx}{z-x} \Rightarrow \rho(x) = \lim_{N\to\infty} R_1(x)$,
  $G(z)$ satisfies $M+1$ order eq. [Müller 02; Burda et al. 10; Götze, Tikhomirov 10; O'Rourke; Soshnikov 11]

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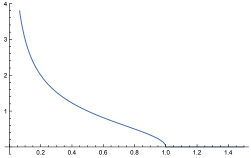
  - soft edge $\boxed{\rho \sim \sqrt{\phantom{x}}}$ vanishing $\forall M \geq 1$



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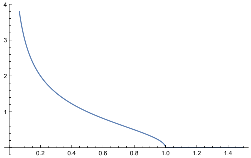


- **local spectrum:**

  - bulk and soft edge **same as for** $M = 1$ [Liu, Wang, Zhang 2016]

  $\boxed{K_{\rm Sine}(x,y) = \frac{\sin(x-y)}{x-y}}$ and

  $\boxed{K_{\rm Airy}(x,y) = \int_0^\infty {\rm Ai}(x+t){\rm Ai}(y+t)dt}$

# . . . local spectrum $M \geq 1$ continued

- $M = 1, 2, \ldots$ **different Meijer-G kernels** ($M = 1$ Bessel)

  [Kuijlaars, Zhang 14]

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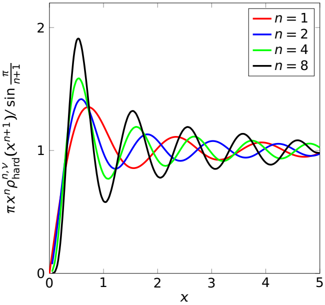


from PhD thesis Ipsen 2014 (unfolded)

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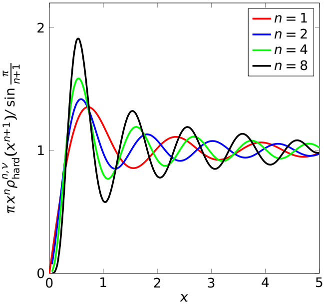


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- local max = 1st, 2nd etc. eigenvalue of $Y^\dagger Y$

$\rightarrow$ for increasing $M$ eigenvalues get more pronounced

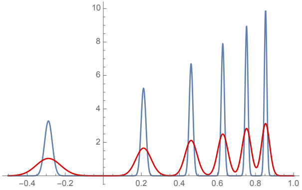
# Limit i): Know results for $M \to \infty$ & limit $N$ fixed

- Eigenvalue density of Lyapunov matrix

$$\rho_L(x) \approx \frac{1}{N}\sum_{j=1}^{N} \frac{1}{\sqrt{2\pi\sigma_j^2}} \exp\left(-\frac{(x-L_j)^2}{2\sigma_j^2}\right) \qquad \text{for} \quad M \gg N$$

$$L_j = \frac{\psi(j)}{2}$$

$$\sigma_j = \sqrt{\frac{\psi'(j)}{4M}}$$




red: $N = 6 \quad M = 100$
blue: $N = 6 \quad M = 1000$

- deterministic values = "picket fence": complex Ginibre [Newman 86; Forrester13], quaternion [Kargin 14] and real [Ipsen 14]

# What can we expect in the double scaling limit iii) $M, N \to \infty$?

- for $M \to \infty$ sufficiently slow still Sine- & Airy-kernel in bulk and at soft edge = **universal correlations** from $M = 1$ cf. [Frahm 95; Ipsen, Schomerus 16]

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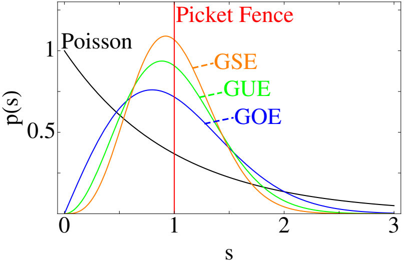
  YES: **linear regime** $\boxed{M = qN}$ with $q$ fixed

# Aside: Level spacing distribution $P(s)$

- popular quantity: $P(s)$ = Fredholm-determinant of $K_{\text{Sine}}$

$\rightarrow$ transition quantum chaos to integrable

[Bohigas,Giannoni,Schmit 84] vs. [Berry, Tabor 77]



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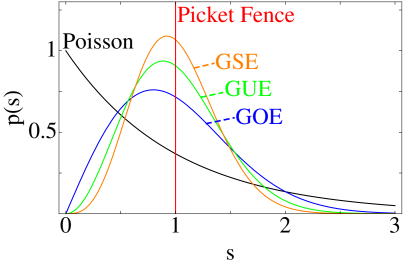


- harmonic oszillator: **equal spacing = picket fence**
  $\Rightarrow \; P(s) = \delta(s-1)$
- Lyapunov exponents not equally spaced $\rightarrow$ unfold:
  $(Y^\dagger Y)^{\frac{1}{M}} = e^{2L}$ has density $\lim_{N\to\infty} R_1(u) = \chi_{[0,1]}$

# Explaining critical scaling $M = qN$

- spacing $\underline{L_{j+1} - L_j} = (\psi(j+1) - \psi(j))/2 \underline{= 1/2j}$
  from $\psi(x) = \log(\Gamma(x))'$ Digamma function
- width $\sigma_j \approx 1/\sqrt{4jM}$

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- **width to spacing ratio** $\boxed{WSR_j = \frac{1}{2}\frac{\sigma_j + \sigma_{j-1}}{L_j - L_{j-1}} \approx \sqrt{j/M}}$

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- **largest exponents** $WSR_N \approx \sqrt{N/M} = 1/\sqrt{q}$, i.e. deterministic for $q \to \infty$ and correlated for $q \ll 1$

# Double scaling: 3 regimes

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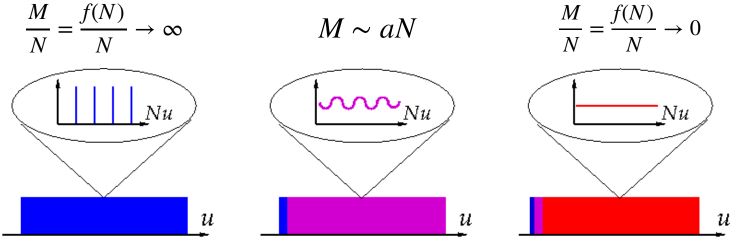


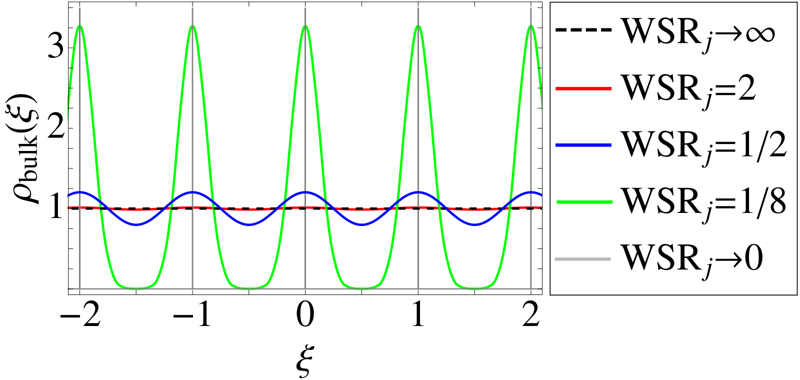

**What are the transition kernels for III) in the middle?**

# Transition regime III): Bulk

- **interpolating bulk kernel:**

$$K_{\rm bulk}(\xi,\zeta;q) \propto \sum_{j=-\infty}^{\infty} e^{j(\xi-\zeta)q}\,\mathrm{Re}\left(\mathrm{erfi}\left[\sqrt{\frac{\pi^2}{2q}} + i\sqrt{q/2}(\zeta-j)\right]\right)$$

- checks: $\lim_{q\to 0} \to K_{\rm Sine}(\xi,\zeta)$ and $\lim_{q\to\infty} \to$ picket fence
- **universality** for coupled Ginibre and Bernoulli (numerics)



# Transition regime III): Soft edge

- **interpolating soft edge kernel:**

$$K_{\rm soft}(\xi,\zeta;q) \propto \int_{i\mathbb{R}+\frac{1}{2}} dt \frac{(1-e^{-tq-(\xi-\zeta)q^{1/3}})^{t-1}}{2\pi i\Gamma[1+t]} e^{t^2-(\gamma(q)+\zeta q^{2/3})t}$$

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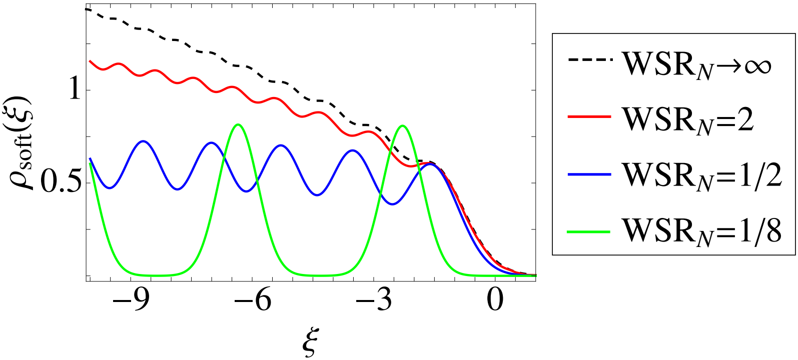
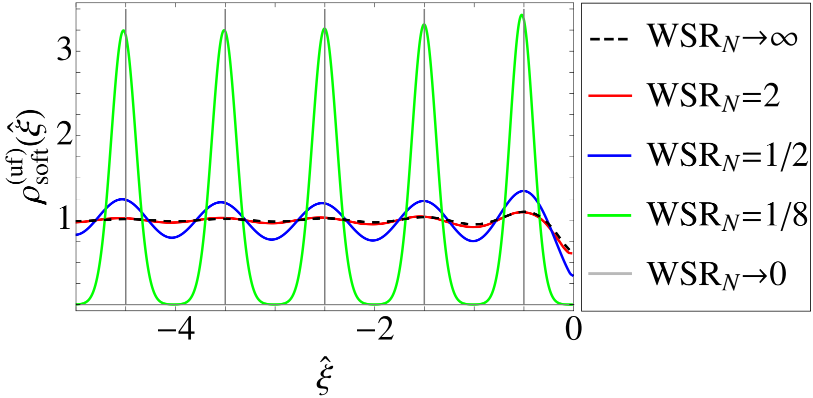


# Soft edge unfolded




$\exists$ similar results for bulk and soft edge with $K \sim \oint\oint$

[Liu, Wang, Wang 1810.00433]

# Summary and some open questions

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  - identified a critical double scaling limit $\boxed{M = qN}$

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  - **2 interpolating kernels:** q-deformed Sine- and Airy-kernel

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**Open**

- transition Tracy-Widom to Gauß for largest Lyapunov exponent

- repeat for complex eigenvalues: stability exponents cf. [Qi et al. 2014-18]

- different products incl. mixed: universality?

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