Eigenfunction distributions for certain random matrix models

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Outlook

1. Rank-one interaction model
2. Rosenzweig-Porter model
3. Power-law random banded and ultrametric matrices

Common point : Variance mixture of resulting distribution

$$P(\Psi) = \int f(y) \frac{1}{\sqrt{2\pi y}} \exp\left(-\frac{|\Psi|^2}{2y}\right) \mathrm{d}y$$

1) E.B. PRL **118**, 022501 (2017)
2) E.B. & M. Sieber, PRE **98**, 032139 (2018)
3) E.B. & M. Sieber, PRE, accepted (2018)

Introduction

- Complex Hamiltonians $\approx$ Random matrix ensembles
- Statistical properties of RME are simple but statistics of eigenvalues and eigenfunctions are complicated
- Universal statistical distributions of eigenvalues (level repulsion)
- For all standard (invariant) ensembles distribution of eigenfunctions is universal (Gaussian or Porter-Thomas) ($x = \sqrt{N}\Psi_j$)

$$P_1(x) = \frac{1}{\sqrt{2\pi l x}} \exp\left(-\frac{x}{2l}\right), \qquad P_2(x) = \frac{1}{l} \exp\left(-\frac{x}{l}\right)$$

$l =$ mean value of x. Standard normalisation : $\langle x \rangle = 1 \longrightarrow l = 1$

C. E. Porter and R. G. Thomas Phys. Rev. **104** 483 (1956)

Fluctuations of nuclear reaction widths

- Proved recently : this is valid for comparable Wigner matrices

$$\frac{C_1}{N} \le \langle H_{ij}^2 \rangle \le \frac{C_2}{N}, \qquad i,j = 1 \dots N$$

Necessity of unusual random matrix ensembles

Old experiments agreed well with PT distribution

Experimental measurements of neutron resonances, (Koehler *et al*)

- *Reduced neutron widths in the nuclear data ensemble : experiment and theory do not agree*, (2011)
- *Neutron resonance data exclude random matrix theory*, (2013)

Investigation of regular graphs with diagonal disorder

- Anderson transition on an infinite random regular tree (Abou-Chacra *et al*, 1973). $W_c \approx 17.5$
- Structure of extended states for RRG. Two conflicting answers
  - The whole extended phase is a metal (Mirlin *et al*, 1996)
  - There is another transition (at $W \approx 10$) from ergodic to non-ergodic phase with non-trivial fractal dimensions (Kravtsov *et al*, 2018)

Rank-one interaction model

- Within standard RME : PT distribution = theorem
- To explain experiments modifications are required

A. Volya, H. A. Weidenmüller, and V. Zelevinsky PRL **115**, 052501 (2015)
Neutron resonance widths and the Porter-Thomas distribution

'Realistic' model of nuclear *s*-wave resonances : $M_{ij} = G_{ij}^{(\beta)} + Z\,\delta_{i1}\delta_{j1}$

$G_{ij}^{(\beta)}$ = standard (GOE or GUE) random matrix
$Z\,\delta_{i1}\delta_{j1}$ = interaction which couples resonances to decay channels

<u>Rank-one formalism</u> : $M_{ij} = G_{ij} + v_i^* v_j$

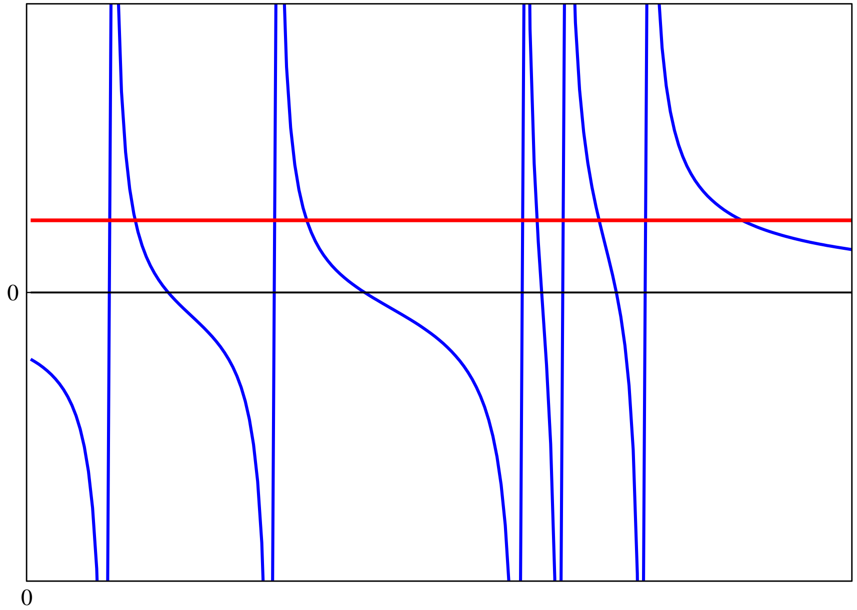
$$\sum_{j=1}^N G_{ij}\Phi_j(\alpha) = e_\alpha \Phi_i(\alpha), \qquad \sum_{j=1}^N M_{ij}\Psi_j(\alpha) = E_\alpha \Psi_i(\alpha)$$

$$\Psi_j(\alpha) = \sum_{\beta=1}^N C_{\alpha\beta}\Phi_j(\beta), \qquad \Phi_j(\alpha) = \sum_{\beta=1}^N C_{\alpha\beta}^{-1}\Psi_j(\beta)$$

$$C_{\alpha\,\beta} = \frac{a_\alpha b_\beta^*}{E_\alpha - e_\beta}, \qquad b_\beta = \sum_{j=1}^N v_j \Phi_j(\beta), \quad a_\alpha = \sum_\beta C_{\alpha\beta} b_\beta$$

Quantisation conditions (E_α, e_α are interlasing)

$$\sum_\beta \frac{|b_\beta|^2}{E_\alpha - e_\beta} = 1, \qquad \sum_\beta \frac{|a_\beta|^2}{e_\alpha - E_\beta} = -1$$



Solve for numerators

$$|b_\alpha|^2 = \frac{\prod_\gamma (E_\gamma - e_\alpha)}{\prod_{\gamma\neq\alpha}(e_\gamma - e_\alpha)}, \qquad |a_\alpha|^2 = -\frac{\prod_\gamma (e_\gamma - E_\alpha)}{\prod_{\gamma\neq\alpha}(E_\gamma - E_\alpha)}$$

Initial probability distribution

Eigenvalues e_α and eigenfunctions $\Phi_1(\alpha)$ of matrix G^β are distributed as in standard random matrix ensembles ($r_\alpha = |\Phi_1(\alpha)|^2$)

$$P(\{e_\alpha\},\{r_\alpha\}) \sim \prod_{\alpha<\gamma} |e_\gamma - e_\alpha|^\beta \prod_\alpha r_\alpha^{\beta/2-1} \delta\left(\sum_\alpha r_\alpha - 1\right) \exp(-V(\{e_\alpha\}))$$

$V(\{e_\alpha\})$ = confinement term. For standard RME : $V(\{e_\alpha\}) = \frac{\beta}{4\sigma^2}\sum_\alpha e_\alpha^2$

Two changes of variables : $(e_\alpha, \Phi_1(\alpha)) \longrightarrow (e_\alpha, E_\alpha) \longrightarrow (\Psi_1(\alpha), E_\alpha)$

New joint distribution = the old one ($r_\alpha = |\Phi_1(\alpha)|^2$, $z_\alpha = |\Psi_1(\alpha)|^2$)

$$\prod_{\alpha<\gamma} |e_\gamma - e_\alpha|^\beta \prod_\alpha r_\alpha^{\beta/2-1} \delta(\sum_\alpha r_\alpha - 1) \prod_\alpha \mathrm{d}e_\alpha \mathrm{d}r_\alpha =$$

$$= \prod_{\alpha<\gamma} |E_\gamma - E_\alpha|^\beta \prod_\alpha z_\alpha^{\beta/2-1} \delta(\sum_\alpha z_\alpha - 1) \prod_\alpha \mathrm{d}E_\alpha \mathrm{d}z_\alpha$$

Symmetry : $G_{ij} = M_{ij} - v_i^* v_j \quad e_\alpha \leftrightarrow -E_\alpha \quad \tilde{P}(\{e_\alpha\},\{E_\alpha\})$ is symmetric

Final answer

Symmetry is valid only **without** the confinement term :

$$\exp(-\beta \mathrm{Tr}\, GG^\dagger/4\sigma^2)$$

Full joint dist of new eigenvalues E_α and new eigenvectors, $z_\alpha \equiv |\Psi_1(\alpha)|^2$

$$P(\{E_\alpha\},\{z_\alpha\}) \sim \prod_{\alpha<\beta} |E_\beta - E_\alpha|^\beta \prod_\alpha z_\alpha^{\beta/2-1} \delta(\sum_\alpha z_\alpha - 1)$$
$$\times \exp\left[-\frac{\beta}{4\sigma^2}(\sum_\alpha E_\alpha^2 - 2Z\sum_\alpha E_\alpha z_\alpha)\right]$$

In large N limit :

Local PTD (=the Gaussian with variance depending on E)

For $-2\sigma\sqrt{N} \le E \le 2\sigma\sqrt{N}$ and all κ, $x = N|\Psi_j(E)|^2$

$$P_\beta(x,E) = \frac{1}{(2\pi x)^{1-\beta/2}(I(E))^{\beta/2}} \exp\left(-\frac{\beta x}{2I(E)}\right)$$

$$I(E) = \left(\kappa^2 + 1 - \frac{\kappa}{\sigma\sqrt{N}}E\right)^{-1}$$

Finite-window distribution

- Local PTD only for $\Psi_1(E)$ in small windows $|\delta E| \ll \sigma\sqrt{N}$
- If $E_1 < E_\alpha < E_2$ the full distribution = weighted integral of local PT distributions (= variance mixture)

$$\mathcal{P}_\beta(x) = \frac{1}{\delta N}\int_{E_1}^{E_2} \frac{\rho_W(E)}{(2\pi x)^{1-\beta/2}(I(E))^{\beta/2}} \exp\left(-\frac{\beta x}{2I(E)}\right) \mathrm{d}E$$

$$I(E) = \frac{1}{\kappa^2 + 1 - \frac{\kappa}{\sigma\sqrt{N}}E}, \qquad \delta N = \int_{E_1}^{E_2} \rho_W(E)\mathrm{d}E$$

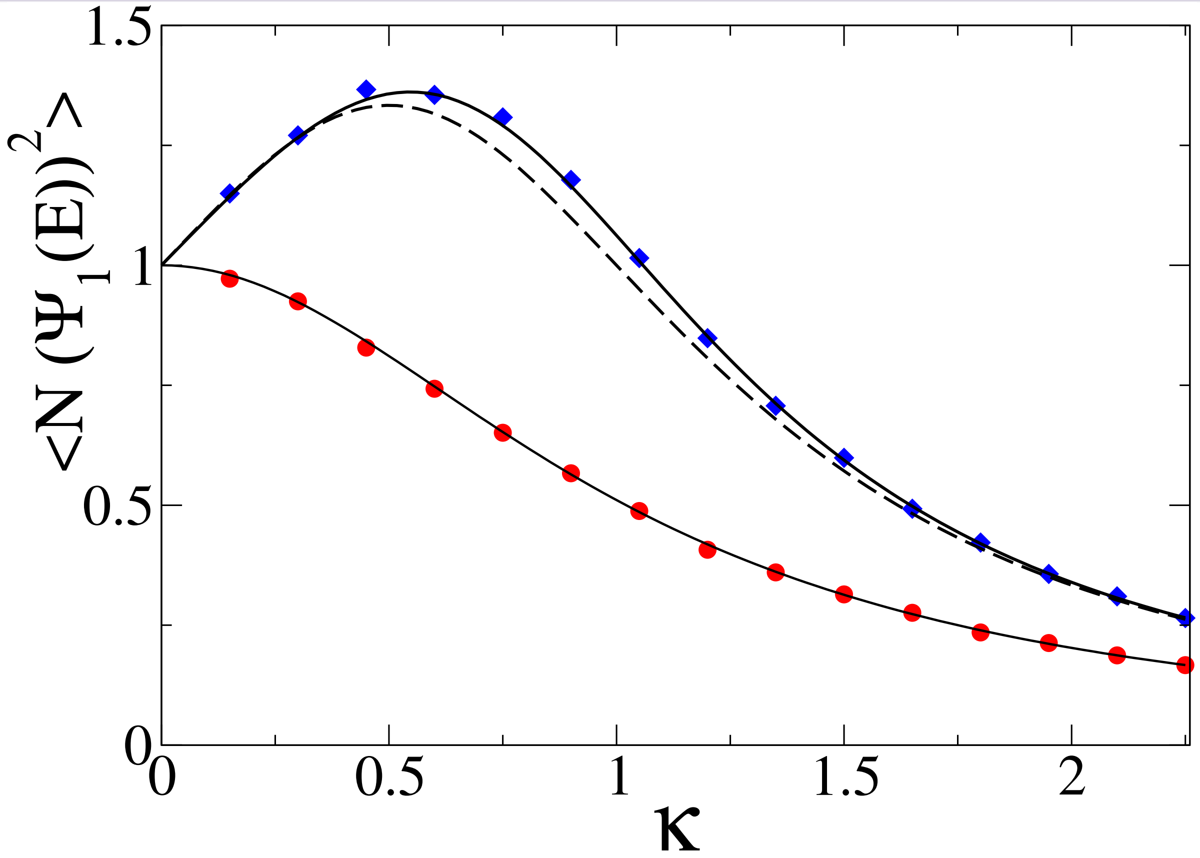
- If all states are included, $E_1 = -2\sigma\sqrt{N}$, $E_2 = 2\sigma\sqrt{N}$
- For $\beta = 1$ (GOE)

$$\mathcal{P}_1(x) = \sqrt{\frac{2}{\pi^3 x}} \int_0^{\pi} \mathrm{d}\phi \, \sin^2\phi \, \sqrt{\kappa^2 + 1 - 2\kappa\cos\phi}\, \mathrm{e}^{-\frac{1}{2}(\kappa^2 - 2\kappa\cos\phi + 1)x}$$

- For $\beta = 2$ (GUE)

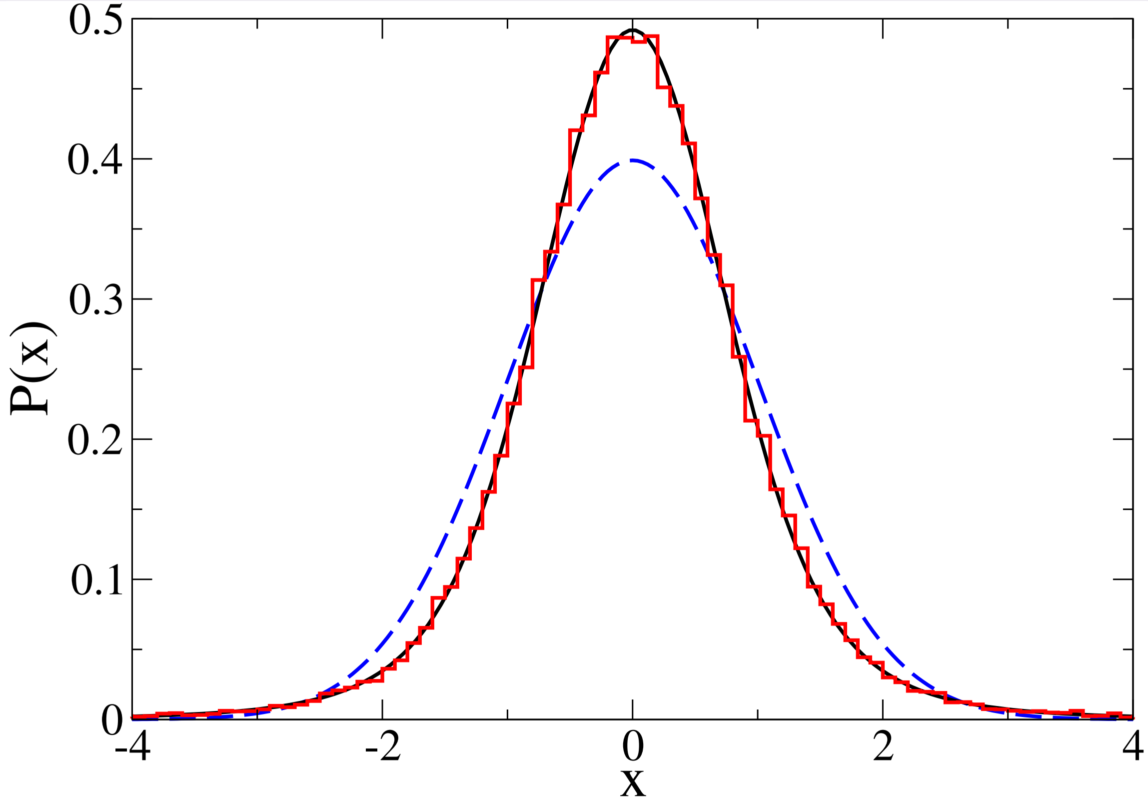
$$\mathcal{P}_2(x) = \frac{I_1(2\kappa x)}{\kappa x}\, \mathrm{e}^{-(\kappa^2+1)x}$$

Numerics : mean values of $N\langle(\Psi_1(E))^2\rangle$ for different κ



- Red circles are mean values for energies in the interval $[-\sqrt{N}/2, \sqrt{N}/2]$
- Blue diamonds are the same but for energies in the interval $[\sqrt{N}/2, 3\sqrt{N}/2]$
- Solid black lines = large-window theoretical predictions
- Dashed black line is the small-window predictions
- $N = 1000$ and each point is averaged over 50 random realisations

Numerics : distribution of $x = \sqrt{N}\Psi_1(E)$ for all states, $\kappa = .8$, $\beta = 1$



- Blue dashed line is the PT distribution (Gaussian) : $P(x) = \mathrm{e}^{-x^2/2}/\sqrt{2\pi}$
- Black solid line is theoretical prediction

$$P(x) = \sqrt{\frac{2}{\pi^3}} \int_0^{\pi} \mathrm{d}\phi \sin^2\phi \sqrt{\kappa^2 + 1 - 2\kappa\cos\phi}\, \mathrm{e}^{-\frac{1}{2}(\kappa^2 - 2\kappa\cos\phi + 1)x^2}$$

Conclusion of the first part

- Standard PT distribution

$$P_1(x) = \frac{1}{\sqrt{2\pi l x}} \exp\left(-\frac{x}{2l}\right), \quad P_2(x) = \frac{1}{l}\exp\left(-\frac{x}{l}\right), \quad x = N|\Psi|^2, \quad l = 1$$

- When ensemble of standard random matrices with Gaussian distribution is perturbed by a rank-one perturbation, the distribution of $x = N|\Psi_1(E)|^2$ has the same functional form but with

$$l(E) \equiv \langle N\,|\Psi_1(E)|^2\rangle = \left(\kappa^2 + 1 - \frac{\kappa}{\sigma\sqrt{N}}E\right)^{-1}$$

- When $\kappa^2 > 1$ there exists one collective state whose mean energy is $E_c = \sigma\sqrt{N}(\kappa + \kappa^{-1})$ and $\langle|\Psi_1^{(c)}|^2\rangle = 1 - \kappa^{-2}$
- When all eigenfunctions in a large energy interval are considered their distribution is not Gaussian but is given by an integral over Gaussian functions (variance mixture)
- In the limit $N \to \infty$ all other components of eigenfunctions (except $\Psi_1(E)$) remain distributed according to the usual PT distribution

Rosenzweig-Porter model (model with fractal eigenvectors)

- Each element is i.i.d. (up to symmetry) Gaussian variables

$$\langle H_{ij}\rangle = 0, \quad \langle H_{ii}^2\rangle = 1, \quad \langle H_{ij}^2\rangle_{i\neq j} = \frac{\epsilon^2}{N^\gamma}, \quad 1\leq i,j \leq N$$

- Define two moments

$$S_1(N) = \frac{1}{N}\sum_{i,j=1}^{N}\langle |H_{ij}|\rangle, \qquad S_2(N) = \frac{1}{N}\sum_{i,j=1}^{N}\langle |H_{ij}|^2\rangle.$$

- The rule of thump
  - If $\lim_{N\to\infty} S_1(N) < \infty \Longrightarrow$ eigenvectors are localised and the spectral statistics is Poissonian
  - If $\lim_{N\to\infty} S_2(N) = \infty \Longrightarrow$ eigenvectors are fully delocalised and the spectral statistics is GOE
- $\gamma > 2 \Longrightarrow$ localisation
- $\gamma < 1 \Longrightarrow$ standard RME

Intermediate region : $1 < \gamma < 2$

Fractal eigenvectors (Kravtsov, 2015)

Rigorously proved (2017)

- Moments of eigenvectors

$$I_q = \langle \sum_j |\Psi_j|^{2q} \rangle \underset{N\to\infty}{\longrightarrow} N^{-\tau_q}, \quad \tau_q = (q-1)D_q$$

- D_q = fractal dimensions
  - Localisation $\Longrightarrow D_q = 0$
  - Delocalised (metal, RMT) $\Longrightarrow D_q = 1$
- RP model $1 < \gamma < 2$: $$\tau(q) = \begin{cases} \gamma q - 1, & q < \frac{1}{2} \\ (q-1)(2-\gamma), & q > \frac{1}{2} \end{cases}$$

Purpose : to find exact distribution of eigenvectors when $1 < \gamma < 2$

Two main ingredients :

1) Breit-Wigner distribution of $\langle |\Psi_j(E)|^2 \rangle$

2) Local Gaussian distribution

Breit-Wigner form of mean square of eigenvectors

$$\Sigma_j^2(E) \equiv \langle |\Psi_j(E)|^2 \rangle \approx \frac{C^2\, \Gamma(E)}{\pi \rho(E) N \left[(E-e_j)^2 + \Gamma^2(E) \right]}$$

- Average is over off-diagonal elements, diagonal elements are fixed
- $\Gamma(E) =$ the spreading width given by the Fermi golden rule

$$\Gamma(E) = \frac{\pi \epsilon^2}{N^{\gamma-1}} \rho_f(E)$$

- $\rho_f(E) =$ level density of final states
- For $N \to \infty$ and $\gamma > 1$ $\rho_f =$ density of diagonal elements

$$\rho_f(E) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{E^2}{2} \right)$$

- C depends on the chosen normalisation
- Standard normalisation

$$\sum_\alpha |\Psi_j(\alpha)|^2 = 1 \longrightarrow \int \rho(E) \langle |\Psi_j(E)|^2 \rangle \mathrm{d}E = \frac{1}{N} \longrightarrow C = 1$$

Recursive relation for the Green function $G=(E-\mathrm{i}\eta-H)^{-1}$

- Identity (Schur's complement formula)

$$G_{ii}(E-\mathrm{i}\eta)=\Big(E-\mathrm{i}\eta-H_{ii}-\sum_{j,k\neq i}H_{ij}G^{(i)}_{jk}(E-\mathrm{i}\eta)H_{ki}\Big)^{-1}$$

- $G(E)^{(i)}=$ the Green function after by removing the row and column i
- For large N

$$\sum_{j,k\neq i}H_{ij}G^{(i)}_{jk}H_{ki}\approx\frac{\epsilon^2}{N^{\gamma-1}}\Big(\frac{1}{N}\sum_{j\neq i}\tilde{G}_{jj}\Big)\approx\frac{\epsilon^2}{N^{\gamma-1}}\Big(\frac{1}{N}\sum_{j\neq i}\frac{1}{E-\mathrm{i}\eta-e_j}\Big)$$

- Self-averaging

$$\frac{1}{N}\sum_{j\neq i}\frac{1}{E-\mathrm{i}\eta-e_j}\underset{N\to\infty}{\longrightarrow}\int\frac{\rho_f(e)\mathrm{d}e}{E-\mathrm{i}\eta-e}$$

- Ignoring real part ($\approx$ small energy shift) and using

$$\mathrm{Im}\;G_{ii}(E-\mathrm{i}\eta)\underset{\eta\to 0}{\longrightarrow}\pi\langle|\Psi_i(E)|^2\rangle\rho(E)$$

one gets the Breit-Wigner expression for $\langle|\Psi_i(E)|^2\rangle$

Local Gaussian distribution (with variance depending on E)

$$P(\Psi_j(E)) = \frac{1}{\sqrt{2\pi\Sigma_j^2(E)}} \exp\left(-\frac{|\Psi_j(E)|^2}{2\Sigma_j^2(E)}\right)$$

Eigenvector distribution

$$P(x)_E = \frac{1}{2\pi\sqrt{a}} \int_{-\infty}^{\infty} \sqrt{(E-e)^2+\Gamma^2(E)}\, \exp\left(-\frac{x^2}{2a}\Big((E-e)^2+\Gamma^2(E)\Big)-\frac{e^2}{2}\right) \mathrm{d}e$$

$$a = \frac{C^2\Gamma(E)}{\pi\rho(E)N} = \frac{C^2\epsilon^2}{N^\gamma}$$

A simple case : distribution in a <u>small</u> window around $E=0$

$$P(x)_{E=0} = \frac{\delta^2}{4\pi\sqrt{a}}\left[K_0(\zeta)+K_1(\zeta)\right]\mathrm{e}^{-\zeta+\frac{\delta^2}{2}}, \quad \zeta = \frac{\delta^2}{4a}(x^2+a), \quad \delta \equiv \Gamma(0) = \frac{\sqrt{\pi}\,\epsilon^2}{\sqrt{2}\,N^{\gamma-1}}$$

Distribution in the bulk and in the tail

- Bulk = x of the order of a

$$y = N^{\gamma/2}\,\Psi_j(E),\quad \langle y^2\rangle = N^{\gamma-1},\quad |y| \le N^{\gamma/2} \longrightarrow C = N^{\gamma/2}, a = \epsilon^2,\ \delta = 0$$

- In the bulk

$$\boxed{P_{\mathrm{bulk}}(y) \approx \frac{\epsilon}{\pi(y^2+\epsilon^2)}}$$

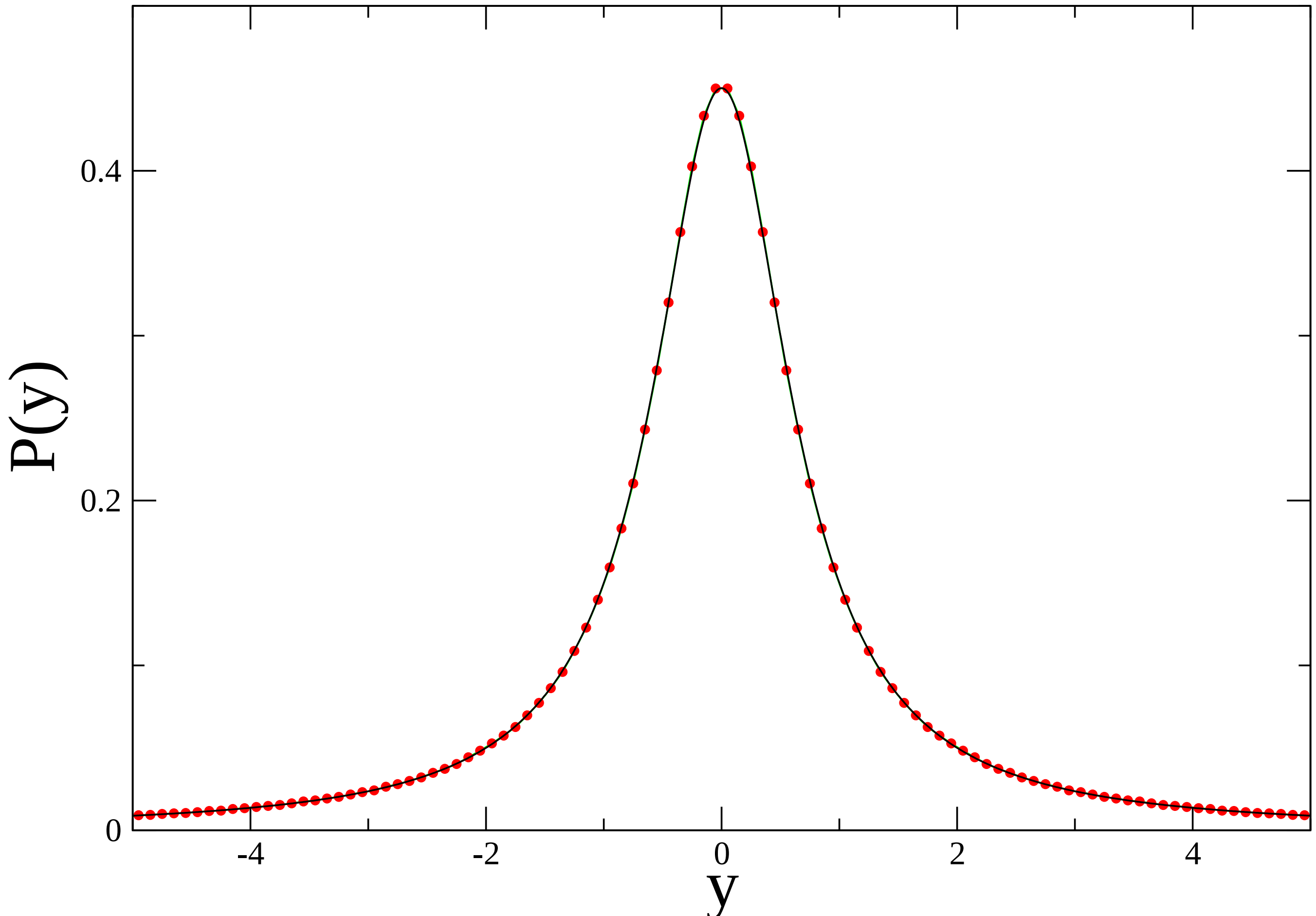
- Tail = large x

$$z = N^{1-\gamma/2}\,\Psi_j(E),\ \langle z^2\rangle = N^{1-\gamma},\ |z| \le N^{1-\gamma/2} \longrightarrow C = N^{1-\gamma/2},\ a = \epsilon^2 N^{2-\gamma}$$

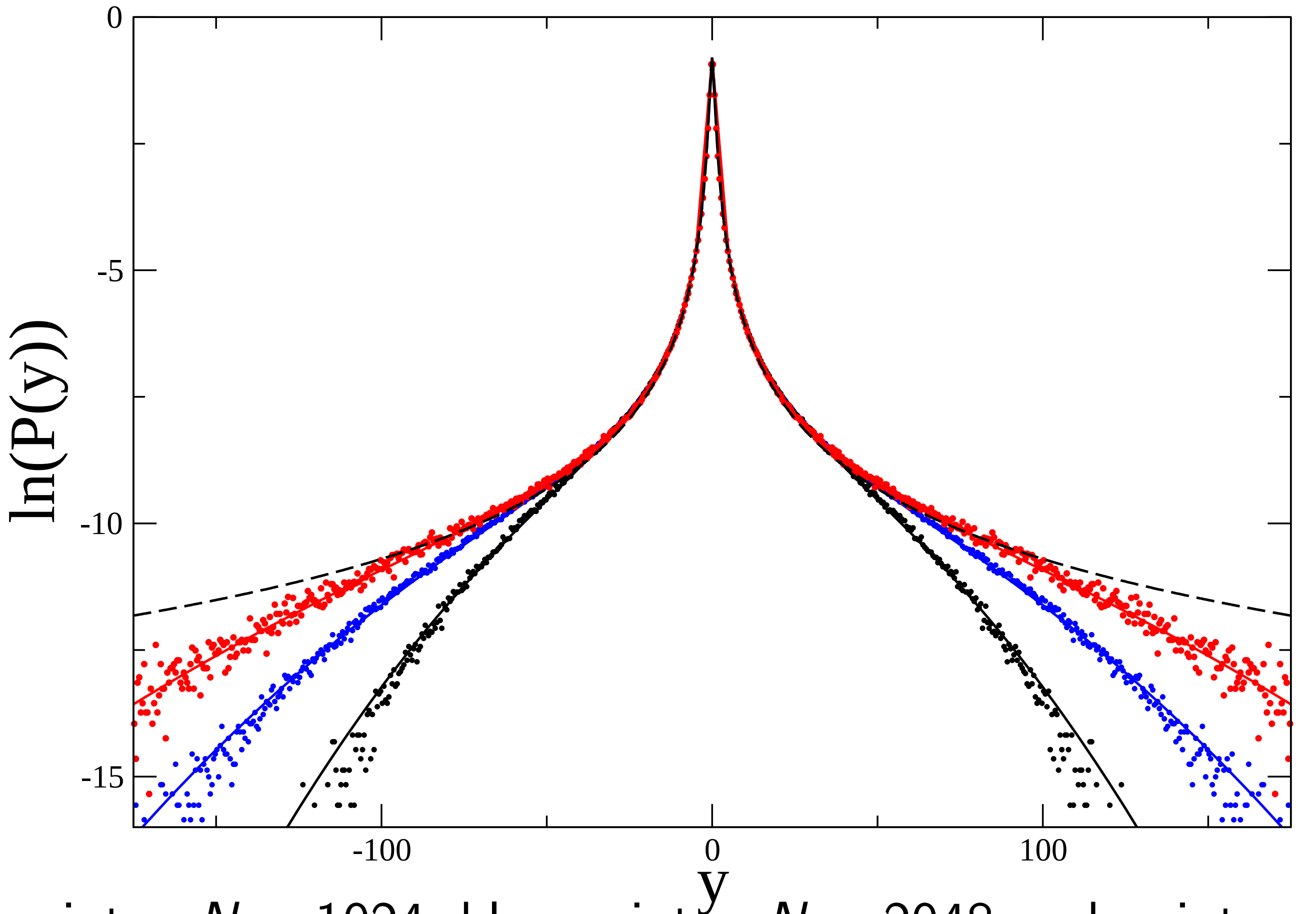
- In the tail

$$\boxed{P_{\mathrm{tail}}(z) = \frac{2\sqrt{2}\,b^3}{\pi\sqrt{\pi}\,N^{\gamma-1}}\left(K_0(b^2z^2)+K_1(b^2z^2)\right)\mathrm{e}^{-b^2z^2}} \qquad b = \frac{\sqrt{\pi}\epsilon}{2\sqrt{2}}$$

Distribution of $y = N^{\gamma/2}\,\Psi_j(E)$ for the RP model with $\gamma = 1.5$, $\epsilon = \frac{1}{\sqrt{2}}$ in the bulk for $N = 4096, 2048, 1024$

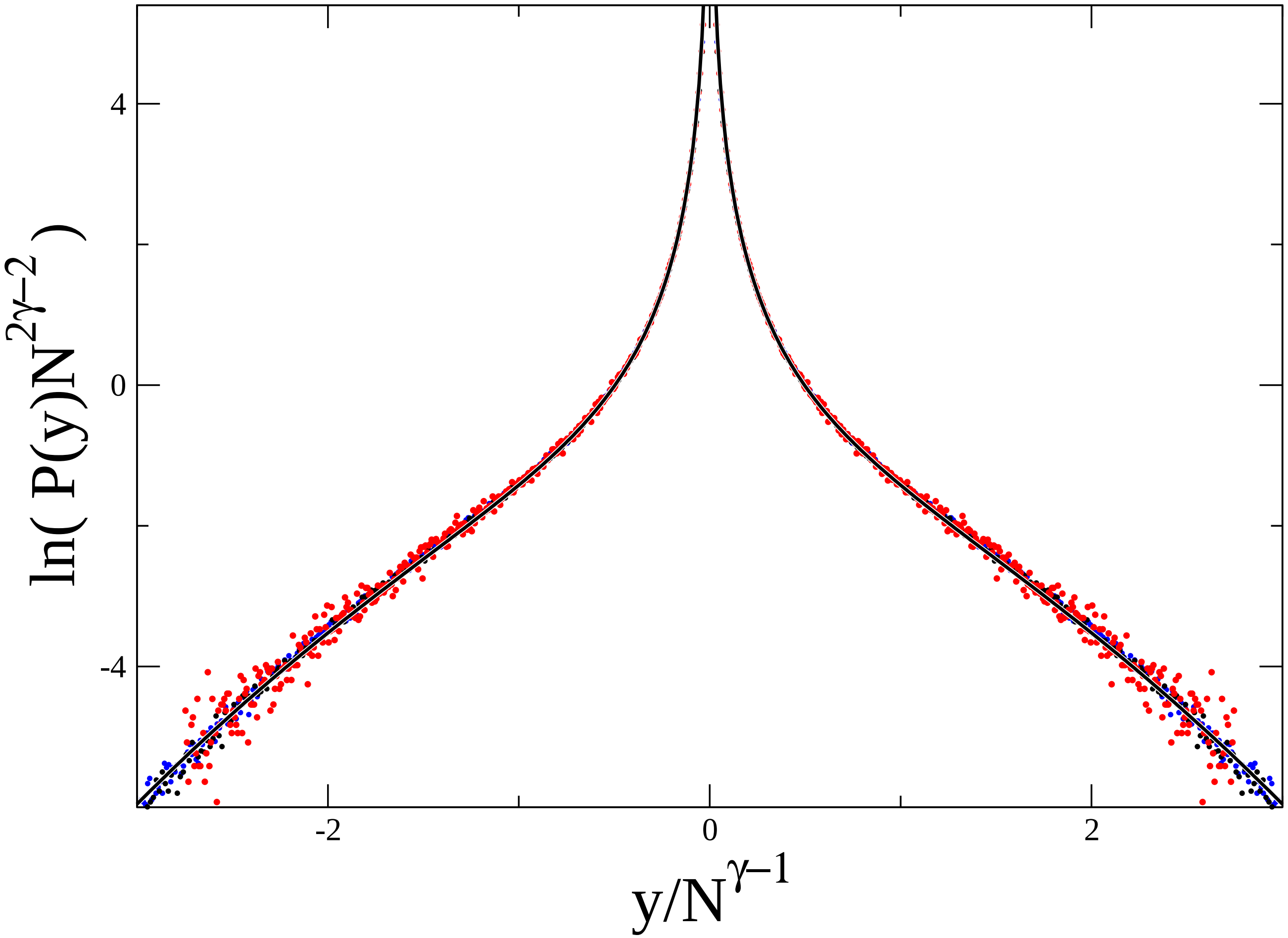


Distribution in logarithmic scale



Black points : $N = 1024$, blue points : $N = 2048$, red points : $N = 4096$
Solid lines of the same colour are theoretical predictions

Distribution in the tail rescaled by $N^{\gamma-1}$



Black points : $N = 1024$, blue points : $N = 2048$, red points : $N = 4096$

Moments in the centre of the spectrum

- Direct calculations

$$I_q \equiv \langle \sum_{j=1}^{N} |\Psi_j(E)|^{2q} \rangle = \frac{2^{q-1/2} a^q \Gamma(q+1/2)}{\sqrt{\pi}\delta^{2q-1}} \Psi\Big(\frac{1}{2}, \frac{3}{2} - q; \frac{\delta^2}{2}\Big)$$

- $\Psi(\alpha, \beta; z) =$ the Tricomi confluent hypergeometric function
- Moments

$$I_{q<\frac{1}{2}} = N^{-\gamma q+1} C_{q<\frac{1}{2}}, \qquad C_{q<\frac{1}{2}} = \frac{\epsilon^{2q}}{\pi} \Gamma(q+1/2)\Gamma(1/2-q) c_{\mathrm{cor}}(q)$$

$$I_{q>\frac{1}{2}} = N^{-(q-1)(2-\gamma)} C_{q>\frac{1}{2}}, \qquad C_{q>\frac{1}{2}} = \frac{\Gamma(q-1/2)\Gamma(q+1/2)}{\pi\, b^{2q-2}\, 2^{q-2}\, \Gamma(q)}$$

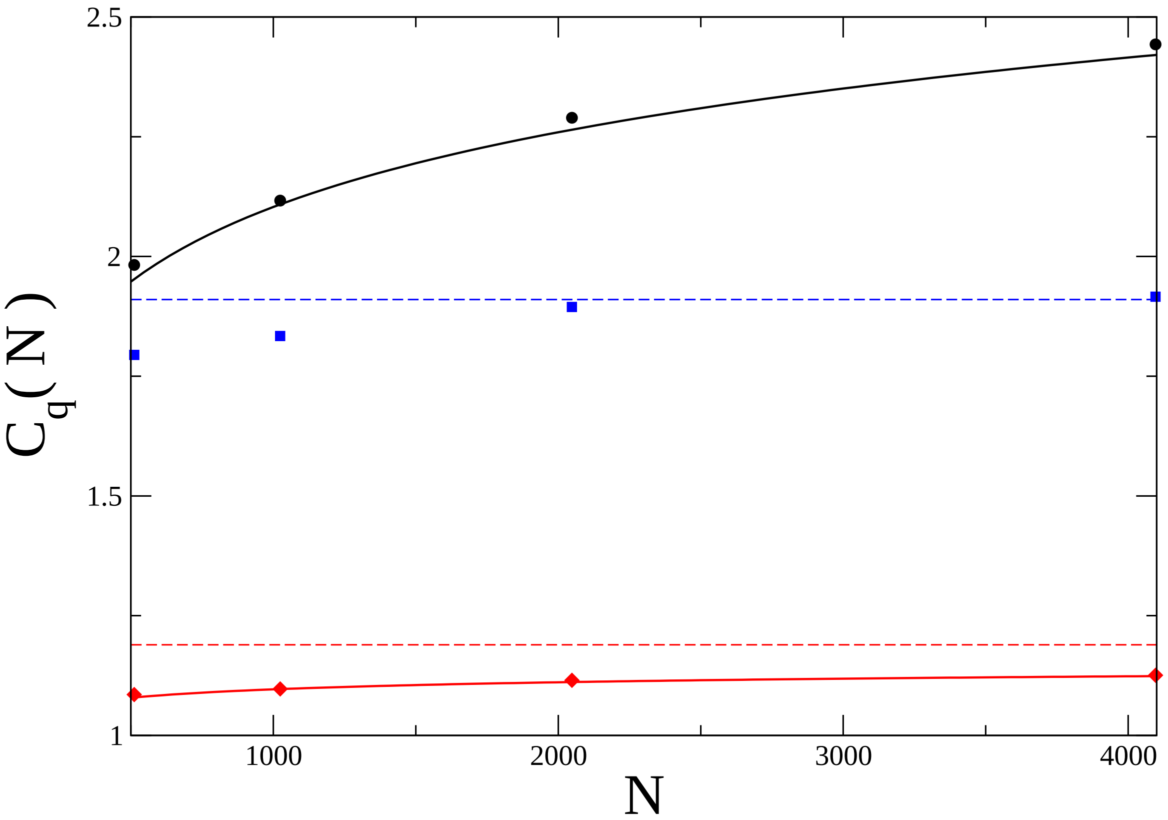
- Correction term $(q<1/2)$

$$c_{\mathrm{cor}}(q) = 1 + \frac{\pi^{1-q}\, \epsilon^{2-4q}\, \Gamma(q-1/2)}{2^{1-2q}\, \Gamma(q)\, \Gamma(1/2-q)} N^{-(\gamma-1)(1-2q)}$$

- The moment $q = \frac{1}{2}$ is unusual

$$I_{\frac{1}{2}} = N^{1-\gamma/2} C_{\frac{1}{2}}, \qquad C_{\frac{1}{2}} = \frac{\epsilon}{\pi}\Big[2(\gamma-1)\ln N - \ln\Big(\frac{\pi\epsilon^4}{16}\Big) - \gamma\Big]$$

Moments versus N



$q = \frac{1}{2}$ (black), $q = 2$ (blue), and $q = \frac{1}{8}$ (red)
Asymptotic values : $C_2 = 1.91,\ C_{\frac{1}{8}} = 1.19$

Red line : the correction term $c_{\mathrm{cor}} \approx 1.19(1 - .44/N^{1/4})$

Conclusion of the second part

- The statistical distribution for eigenfunctions of the Rosenzweig-Porter model in the regime $1 < \gamma < 2$ has been obtained
- Calculations are based on two well accepted and robust physical assumptions
- The first : mean square modulus of eigenfunctions is given by the Breit-Wigner formula with the spreading width calculated by the Fermi golden rule
- The second : eigenfunctions are distributed according to the local Porter-Thomas law with variance given by the above formula
- Result = variance mixture
- Many quantities can be calculated analytically

Note : convergent sums of random numbers

Divergent sums of iid : $X=\frac{1}{n}\sum_{i=1}^{n} x_i \underset{CLT}{\longrightarrow}$ Gaussian distribution

Convergent sums of iid : no general results, large variety of distributions

Bernoilli convolution

$$X_\lambda=\sum_{n=0}^{\infty}\pm\lambda^n, \qquad |\lambda|<1, \qquad \nu_\lambda(E)=Prob(X_\lambda\in E)$$

- $0<\lambda<1/2 \longrightarrow \nu_\lambda$ is the Cantor set of dimension $\ln(2)/\ln\lambda^{-1}$
- $\lambda=1/2 \longrightarrow \nu_{1/2}=$ uniform measure on $[-2,2]$
- $1/2<\lambda<1$ support $[-(1-\lambda)^{-1},(1-\lambda)^{-1}]$
- Th. : For almost all $\lambda\in(1/2,1)$ ν_λ is absolutely continuous
- Th. : If $\frac{1}{\lambda}$ is a Pisot number $\in(1,2)$ then ν_λ is singular (fractal)

Pisot numbers : $\theta>1=$ root of algebraic eq with integer coefficients
All other roots are inside the unit circle
$\theta^2-\theta-2=0$, $\theta_1=(1+\sqrt{5})/2\approx1.618$, $\theta_2=(1+\sqrt{5})/2\approx-0.618$
$\theta^3-\theta-1=0$, $\theta_1\approx1.325$, $\theta_{2,3}\approx-.662\pm0.562i$ $|\theta_{2,3}|\approx0.869$

Random matrix models with state hierarchy

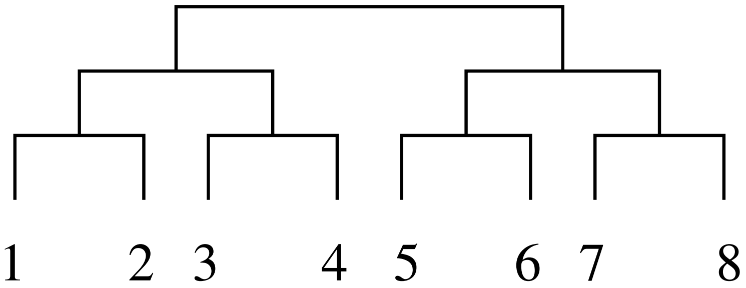
- **Power-law banded matrices** : Each matrix element = independent (up to the symmetry) Gaussian with zero mean and the variance

$$\langle H_{ii}^2 \rangle = 2, \qquad \langle H_{ij}^2 \rangle_{i\neq j} = a^2(|i-j|)$$

- $a(r)$, $r=|i-j|$, decreases as a power of the distance $a(r) \underset{r\to\infty}{\longrightarrow} \epsilon\, r^{-s}$
- E.g. translation-invariant function

$$a(r) = \epsilon \left[1 + \left(\frac{N}{\pi} \sin(\frac{\pi r}{N}) \right)^2 \right]^{-s/2}, \quad a(r) \underset{r\ll N}{\longrightarrow} \frac{\epsilon}{(1+r^2)^{s/2}} \; .$$

- **Ultrametric matrices** : $2^n \times 2^n$ matrices with $a(i,j) = \epsilon\, 2^{-s\, \mathrm{dist}(i,j)}$
- $\mathrm{dist}(i,j)$ = ultrametric distance on a binary tree



$\mathrm{dist}(1,2) = 1$, $\mathrm{dist}(1,3) = \mathrm{dist}(1,4) = 2$,
$\mathrm{dist}(1,5) = \mathrm{dist}(1,6) = \mathrm{dist}(1,7) = \mathrm{dist}(1,8) = 3$

Localisation-delocalisation transition (as above)

- Two moments

$$S_1(N) = \frac{1}{N}\sum_{i,j=1}^{N}\langle|H_{ij}|\rangle, \qquad S_2(N) = \frac{1}{N}\sum_{i,j=1}^{N}\langle|H_{ij}|^2\rangle.$$

- The rule of thump
  - If $\lim_{N\to\infty} S_1(N) < \infty \Longrightarrow$ eigenvectors are localised and the spectral statistics is Poissonian
  - If $\lim_{N\to\infty} S_2(N) = \infty \Longrightarrow$ eigenvectors are fully delocalised and the spectral statistics is GOE
- $s > 1 \Longrightarrow$ localisation
- $s = 1 \Longrightarrow$ fractal eigenfunctions
- $s < \frac{1}{2} \Longrightarrow$ standard RME

Intermediate region (the only place for non-ergodic (fractal) behaviour)

$$\frac{1}{2} < s < 1$$

Absence of analytical results

Main numerical results

- No indication of new phases when $\frac{1}{2} < s < 1$. Distribution of $x = \sqrt{N}\Psi_j$ becomes quickly independent of N
- Eigenvector distribution is extremely well approximated by the generalised hyperbolic distribution

$$P_{\mathrm{GHD}}(x) = \frac{\sqrt{\alpha}}{\sqrt{2\pi}\delta^{\lambda}K_{\lambda}(\alpha\delta)}(x^2+\delta^2)^{(\lambda-1/2)/2}\,K_{\lambda-1/2}\left(\alpha\sqrt{x^2+\delta^2}\right)$$

GHD = the variance mixture

$$P_{\mathrm{GHD}}(x) = \int_0^{\infty} P_{\mathrm{GIG}}(y)\,\frac{\mathrm{e}^{-x^2/2y}}{\sqrt{2\pi y}}\,\mathrm{d}y$$

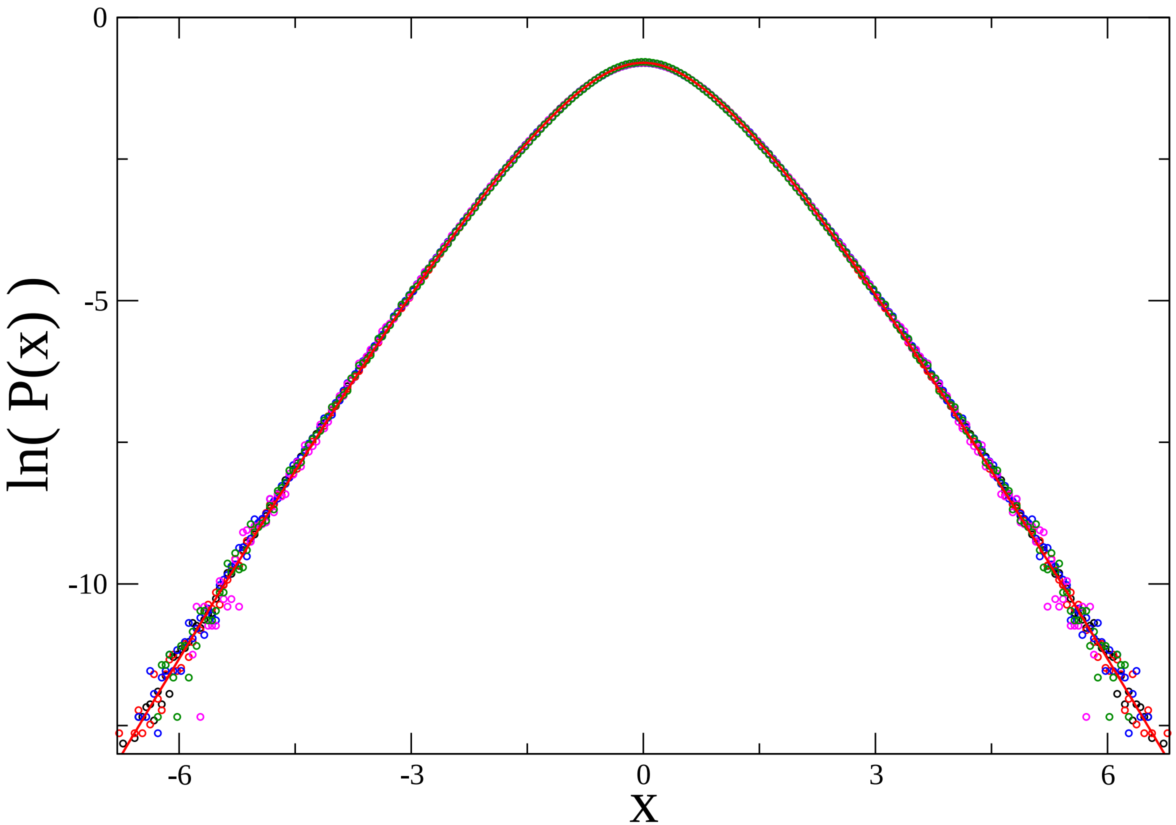
GIG = generalised inverse Gaussian distribution

$$P_{\mathrm{GIG}}(x) = \frac{\alpha^{\lambda}}{2\delta^{\lambda}K_{\lambda}(\alpha\delta)}\,x^{\lambda-1}\mathrm{e}^{-\frac{1}{2}(\alpha^2 x+\delta^2 x^{-1})}\ .$$

Moments : $$C_q \equiv \langle x^{2q}\rangle_{\mathrm{GHD}} = C_{\mathrm{GOE}}(q)\langle x^q\rangle_{\mathrm{GIG}}$$

$$C_{\mathrm{GOE}}(q) = \frac{2^q\Gamma(q+\frac{1}{2})}{\sqrt{\pi}},\qquad \langle x^q\rangle_{\mathrm{GIG}} = \left(\frac{\delta}{\alpha}\right)^q\frac{K_{\lambda+q}(\alpha\delta)}{K_{\lambda}(\alpha\delta)}$$

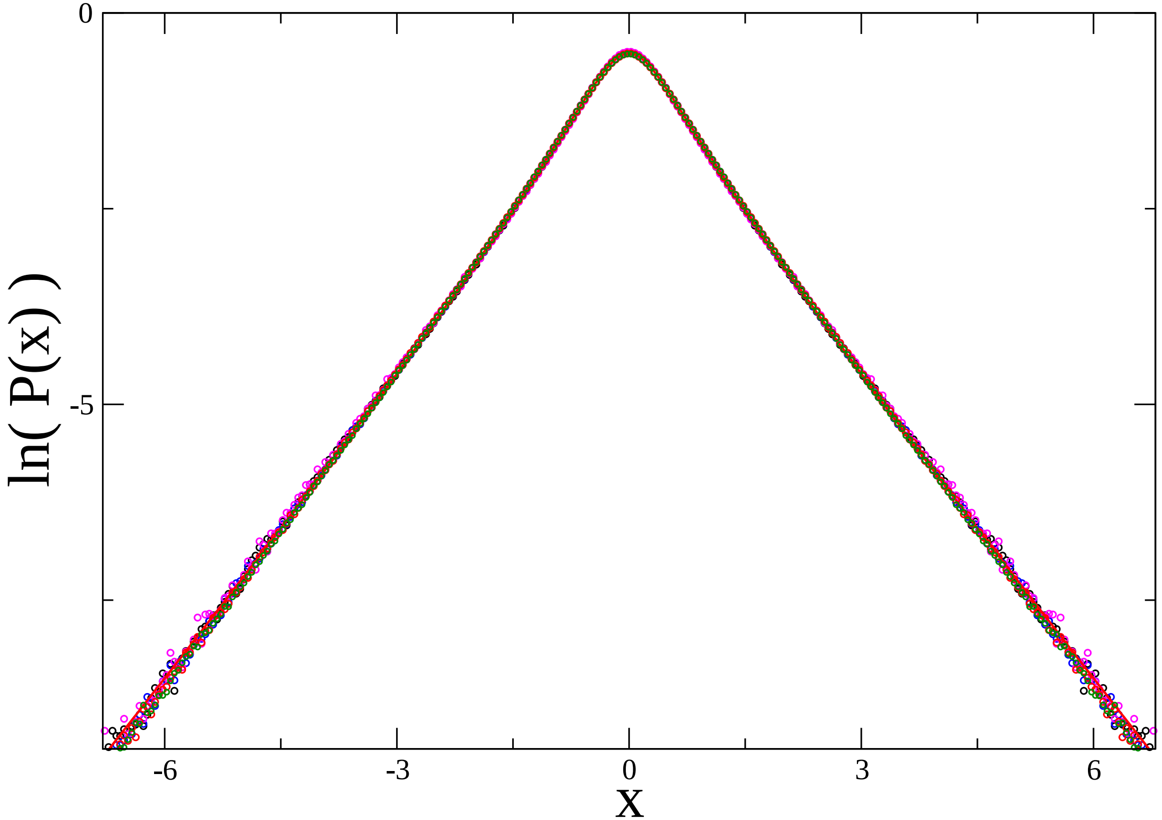
PLBM with $s = 0.7$ and $\epsilon = 1$



Black : $N = 8192$, red : $N = 4096$, blue : $N = 2048$, green : $N = 1024$, magenda : $N = 512$

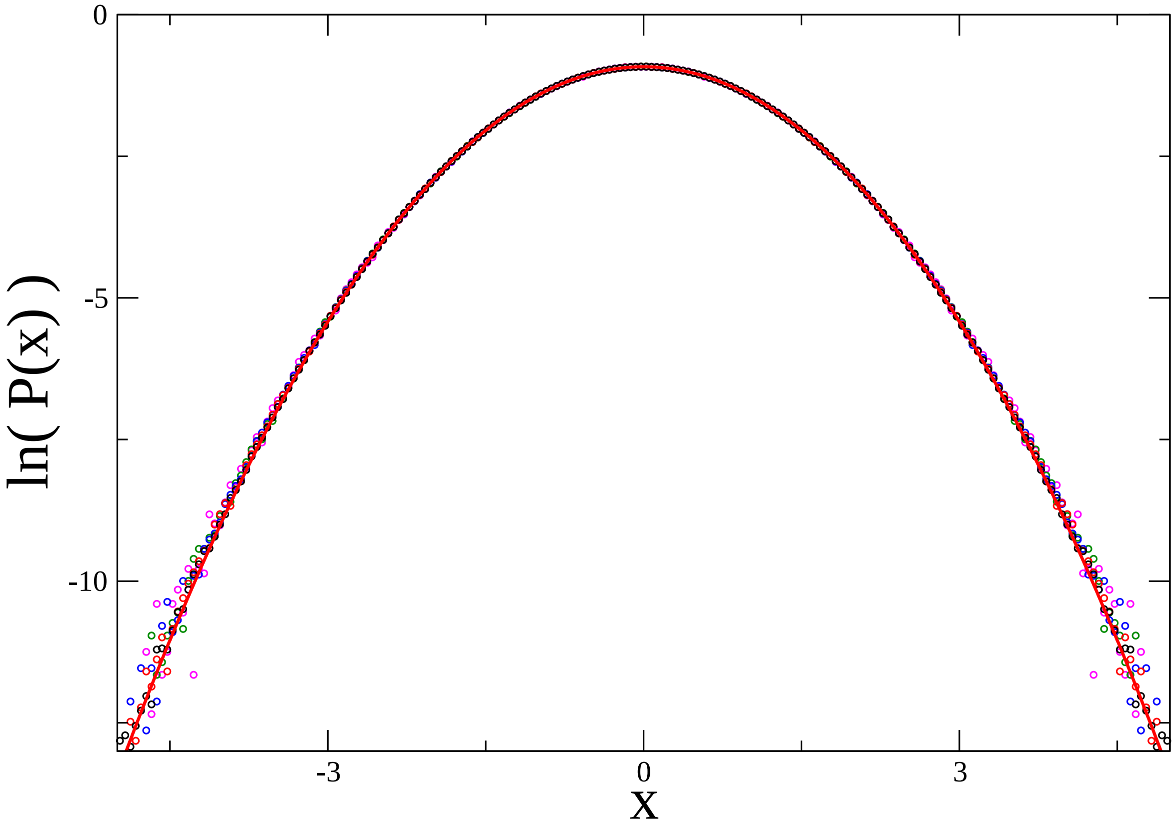
Fit GHD with $\alpha = 2.6154$, $\lambda = 3.3615$, $\delta = 0.2903$

UMM with $s = 0.7$ and $\epsilon = 1$



Black : $N = 8192$, red : $N = 4096$, blue : $N = 2048$, green : $N = 1024$, magenda : $N = 512$
Fit GHD : $\alpha = 1.1673$, $\lambda = 0.3880$, $\delta = 0.4409$

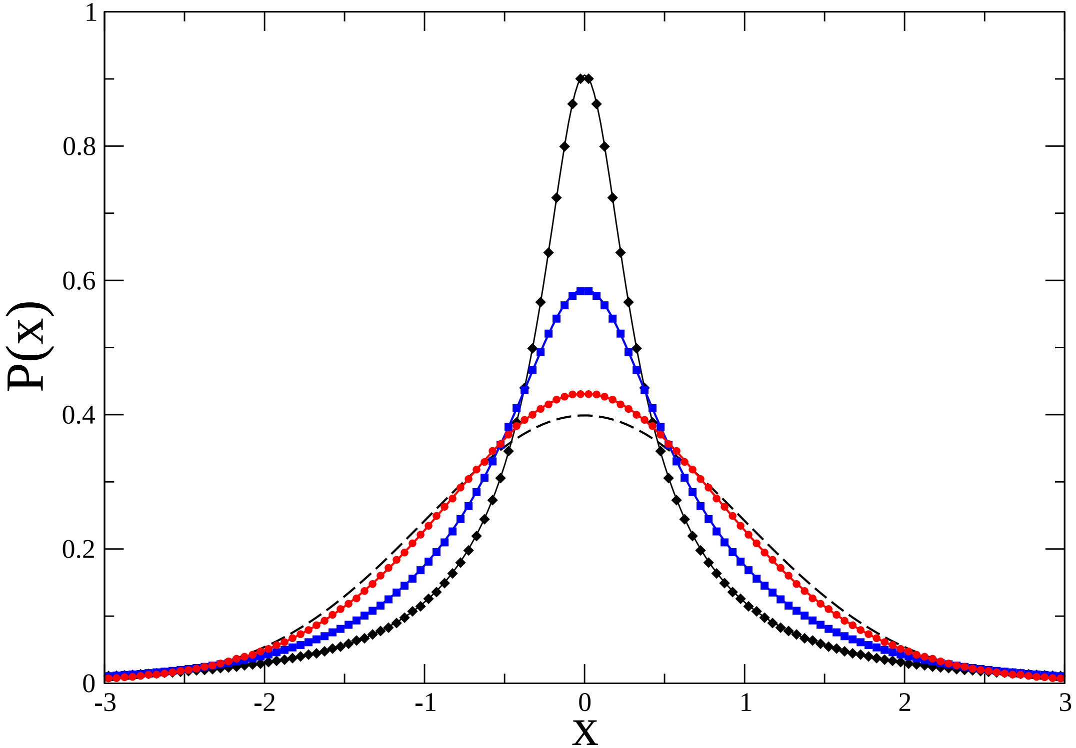
PLBM with $s = 0.3$ and $\epsilon = 1$ (GOE)



Black : $N = 8192$, red : $N = 4096$, blue : $N = 2048$, green : $N = 1024$, magenda : $N = 512$

Solid red line : Porter-Thomas distribution (Gaussian)

PLBM with $s = 0.7$ and different ϵ



$\epsilon = 0.3$ (black circles), $\epsilon = 0.5$ (blue squares), and $\epsilon = 1.5$ (red diamond)

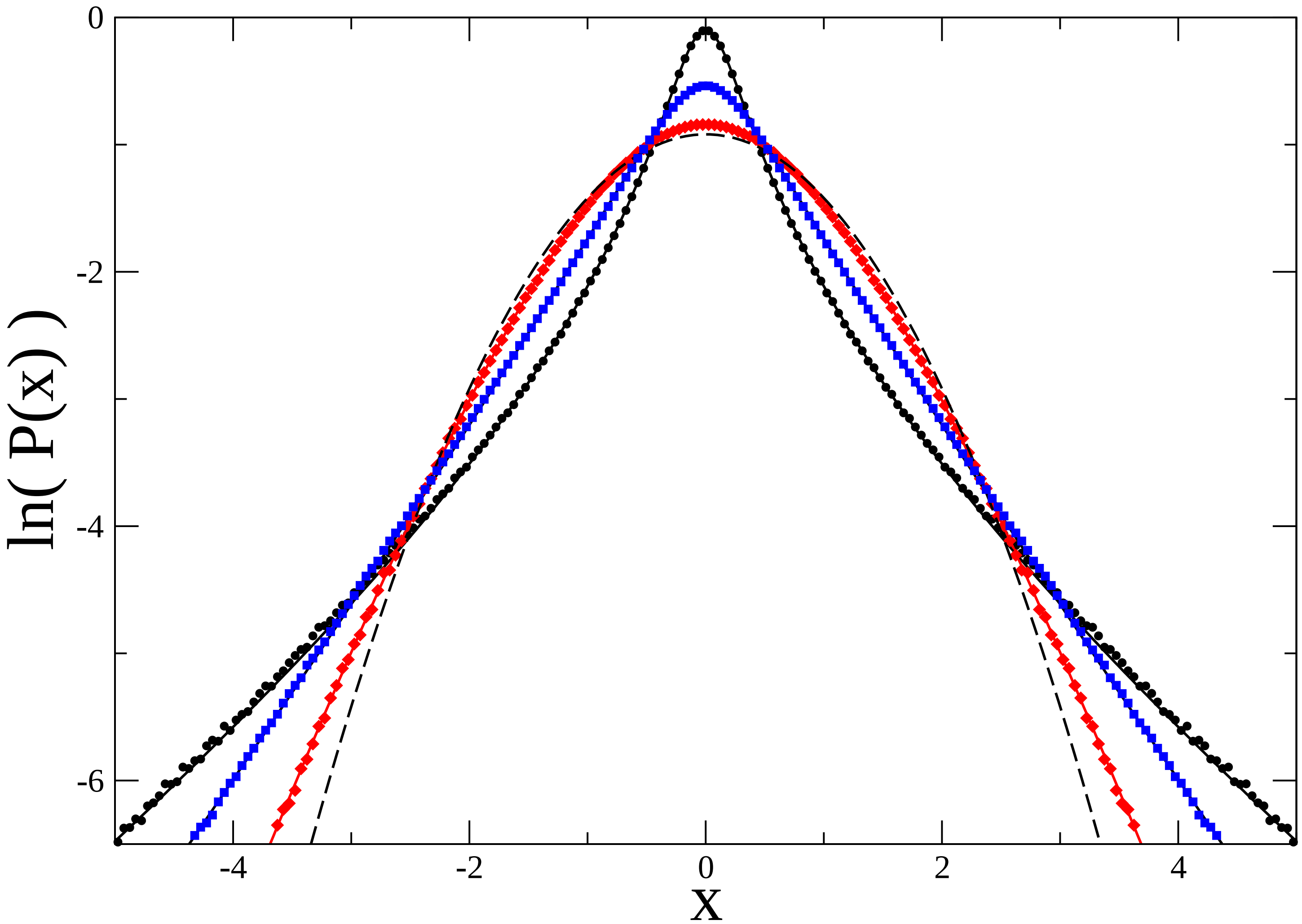
GHD fits :

$\epsilon = 0.3,\ \alpha = 0.6506,\ \lambda = -0.1067,\ \delta = 0.2805$

$\epsilon = 0.5,\ \alpha = 1.2754,\ \lambda = 0.5862,\ \delta = 0.3945$

$\epsilon = 1.5,\ \alpha = 2.9341,\ \lambda = 3.6392,\ \delta = 1.0377$

PLBM with $s = 0.7$ and different ϵ in logarithmic scale

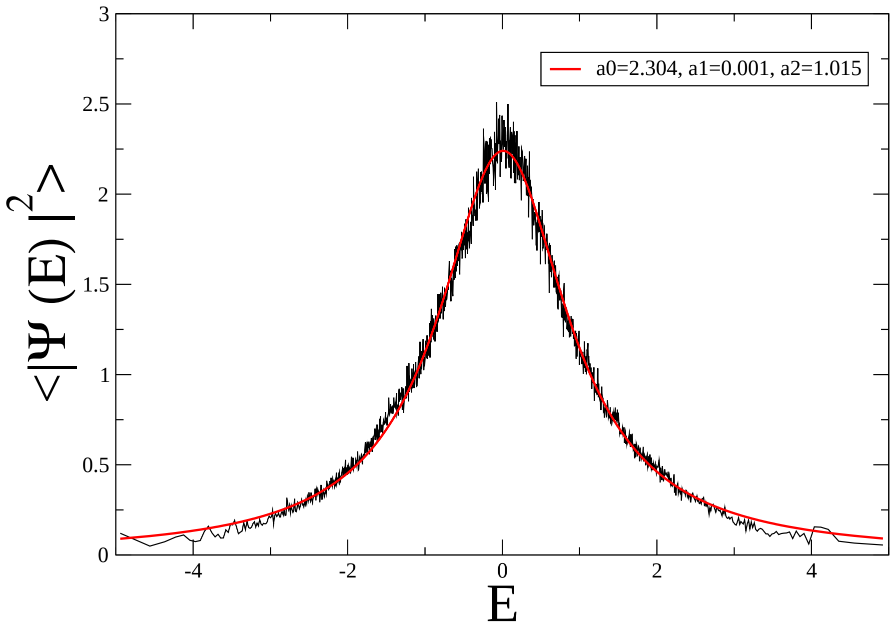


Eigenvector variance averaged over all but one H_{ij}

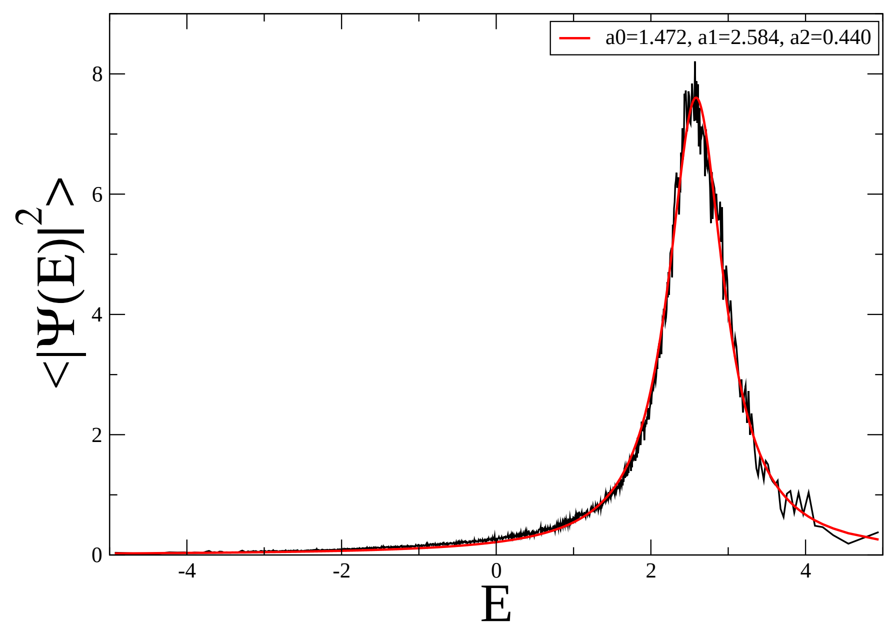
Mean square of eigenfunctions $\Psi_1(E)$ with fixed H_{11} averaged over all other H_{ij}

$$\langle|\Psi_1(E)|^2\rangle = \frac{a_0}{(E-a_1)^2+a_2^2}\,, \qquad a_1 \sim H_{11}$$

$$a_1 + \mathrm{i}a_2 = H_{11} + \sum H_{1k}\, a(|1-k|)\, G_{kj}(E)\, a(|j-1|)\, H_{j1}, \qquad a(|1-k|) \underset{k\to\infty}{\longrightarrow} k^{-s}$$



$H_{11} = 0$



$H_{11} = 2$

Eigenvector variance averaged over a window of energies

- M_I consecutive levels in $I = [E - \delta E/2, E + \delta E/2]$
- Calculate

$$x = \frac{1}{M_I} \sum_{E_\alpha \in I} N |\Psi_i(E_\alpha)|^2$$

- Distribution of x
- If $\Psi_i(E_\alpha)$ are independent (GOE) then $P(x) = \chi^2$ dist with M_I dof

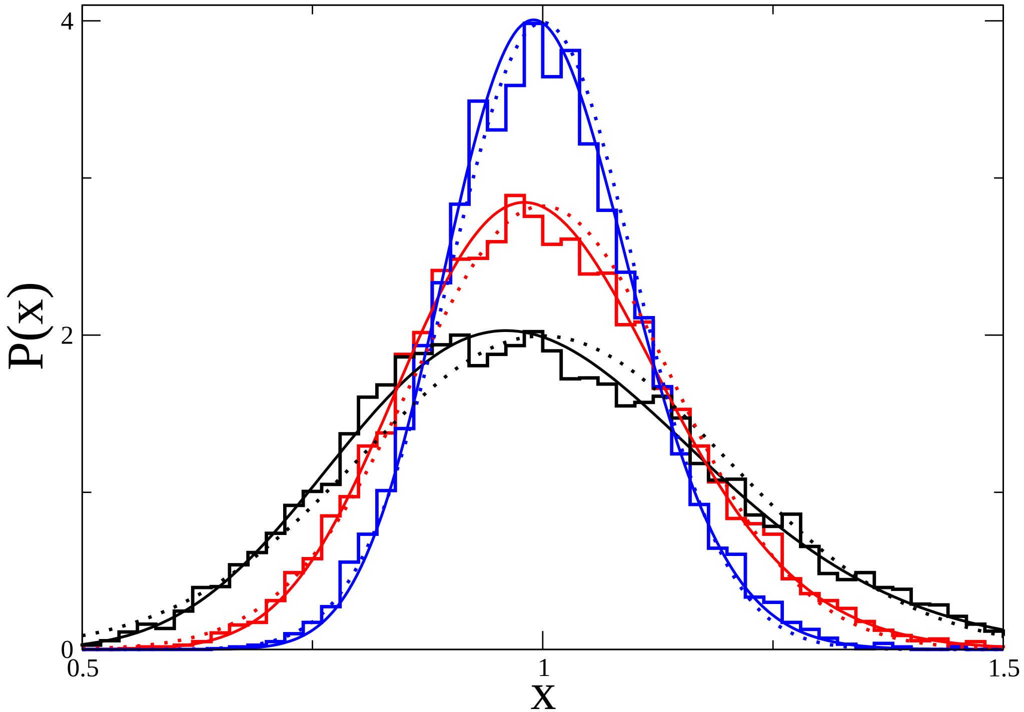
$$P_{\chi^2}(x, \nu) = \frac{\nu^\nu \, x^{\nu/2-1}}{2^{\nu/2}\Gamma(\nu/2)} \mathrm{e}^{-\nu\, x/2}, \qquad \langle x \rangle_{\chi^2} = 1 \; .$$

- When $M_I \to \infty$ asymptotic formulas (central limit theorem)

$$P(x)_{\mathrm{GOE}} \underset{M_I \to \infty}{\longrightarrow} \sqrt{\frac{M_I}{4\pi}} \mathrm{e}^{-M_I (x-1)^2/4}$$

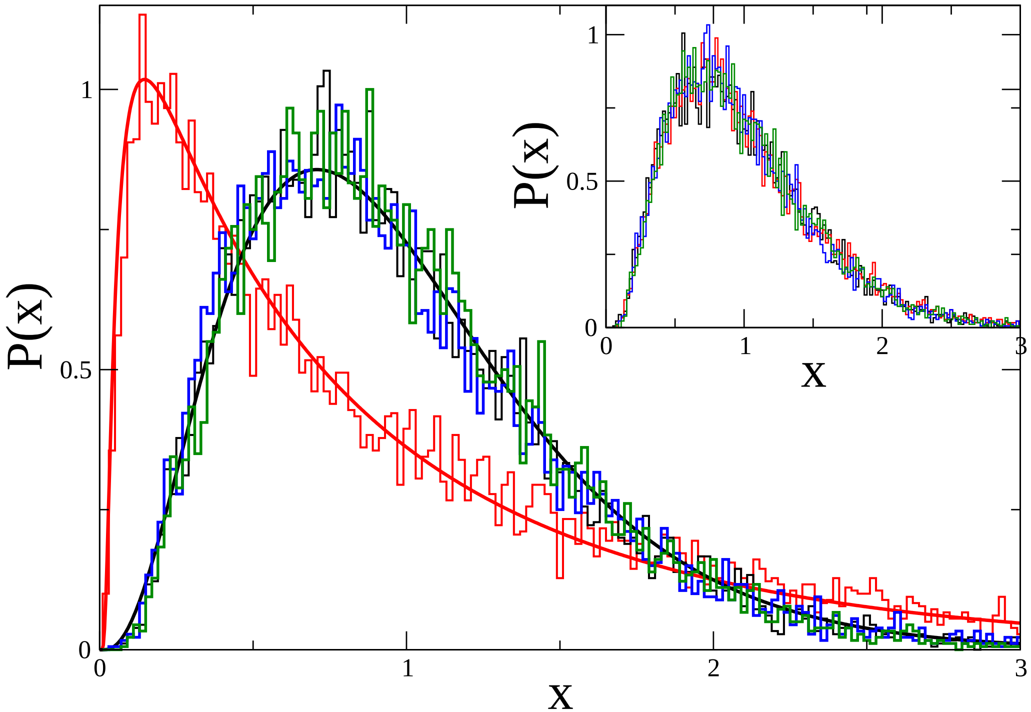
- Deviation from χ^2 distribution $\longrightarrow$ $\Psi_i(E_\alpha)$ are not independent $\approx$ absence of self-averaging

PLBM with $s = 0.3$, $\epsilon = 1$ (GOE), $N = 4096$, and different M_l



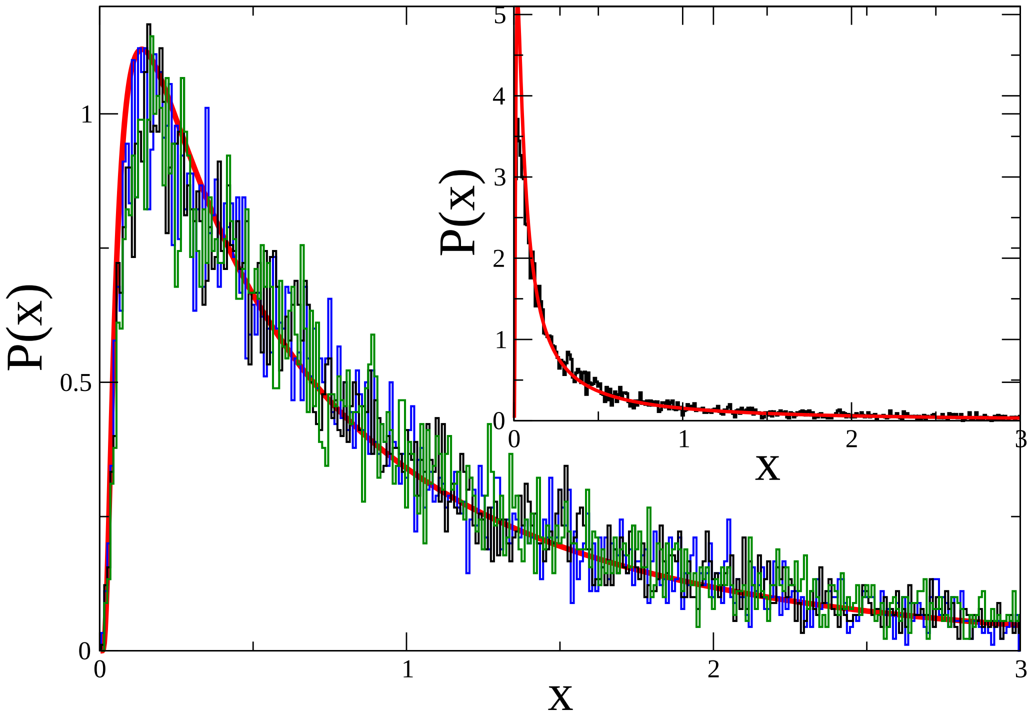
Staircase lines : $M_l = 200$ (blue), $M_l = 100$ (red), $M_l = 50$ (black)
Solid lines = χ^2 distributions with $\nu = M_l$ dof

PLBM with $s = 0.7$, $N = 4096$



Red staircase line : $s = 0.7$, $\epsilon = 0.5$, $M_l = 100$
Other staircase lines correspond to $s = 0.7$ and $\epsilon = 1$ for different M_l
blue : $M_l = 50$, black : $M_l = 100$, green : $M_l = 200$
Solid lines = the GIG distribution. Insert : $s = 0.7$, $\epsilon = 1$, $M_l = 100$ with different $N = 1024, 2048, 4096, 8192$

UMM with $s = 0.7$, $\epsilon = 1$, $\epsilon = 0.5$, $N = 4096$



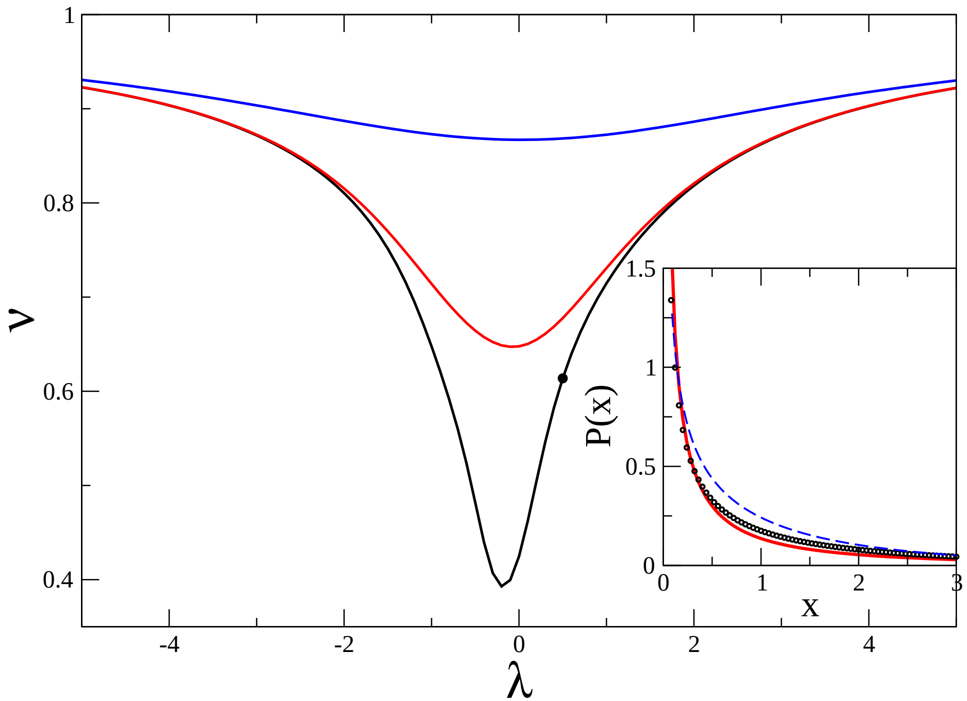
Staircase lines : $M_l = 50, 100, 200$. Solid red line = the GIG distribution
Insert : UMM for $N = 4096$, $s = 0.7$, $M_l = 100$ but for $\epsilon = 0.5$

Possible comparison with experiments

- Experimental results for neutron widths had been fitted by the χ^2 dist :

$$\boxed{P_{\chi^2}(x,\nu) = \frac{\nu^{\nu/2}\, x^{\nu/2-1}}{2^{\nu/2}\Gamma(\nu/2)} \mathrm{e}^{-\nu x/2}} \qquad \langle x \rangle_{\chi^2} = 1 \ .$$

- $^{192}\mathrm{Pt}$: $\nu = 0.57 \pm 0.16$, $^{194}\mathrm{Pt}$: $\nu = 0.47 \pm 0.19$, $^{196}\mathrm{Pt}$: $\nu = 0.60 \pm 0.28$
- The normalised GHD depends on 2 parameters λ and $\xi = \alpha\delta$
- We fix $\xi = 0.02, 0.2, 2$ and for different λ found the best non-linear χ^2 fit



Blue : $\xi = 2$
Red : $\xi = 0.2$
Black : $\xi = 0.02$

Conclusion of the third part

- Power-law banded and ultrametric matrices are typical representatives of random matrix ensembles with varying strength of interaction
- If the interaction decays quickly ($s > 1$) eigenfunctions are localised and eigenvalues behave as iid random variables (Poisson)
- When $s < \frac{1}{2}$ eigenfunctions are fully delocalised and the spectral statistics is the standard RME (GOE)
- In the intermediate region $\frac{1}{2} < s < 1$ eigenfunctions are supposed to be delocalised but their properties remain elusive
- Eigenvector distribution for PLBM and UMM has been calculated numerically for various combinations of model parameters
- No anomalous scaling was observed. After rescaling by $\sqrt{N}$ eigenvector distributions become N-independent functions
- Main result : the eigenvector distributions can be extremely accurately fitted by the generalised hyperbolic distribution (variance mixture)
- The investigation of the PLBM and UMM in the intermediate regime seems to be overlooked but is of importance as they constitute a new class of random matrices potentially important for different applications