Operator-valued zeta functions

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Random Matrices, Integrability and Complex Systems
3 – 8 October 2018

0. Hilbert-Pólya vs operator-valued zeta functions

In the Hilbert-Pólya programme one investigates operators whose eigenvalues correspond to the locations of the zeros of the zeta function.

For example, one might consider the properties of the operator

$$\tfrac{1}{2}(1 - \mathrm{i}\hat{h}_{\rm BK}),$$

where $\hat{h}_{\rm BK} = \hat{x}\hat{p} + \hat{p}\hat{x}$ denotes the Berry-Keating Hamiltonian.

Here we consider the operator

$$\zeta\left(\tfrac{1}{2}(1 - \mathrm{i}\hat{h}_{\rm BK})\right).$$

We investigate such an operator by letting it act on trigonometric functions.

We will find, for $x \in (0, \pi]$, that

$$\zeta\left(\tfrac{1}{2}(1 - \mathrm{i}\hat{h}_{\rm BK})\right) \sin x = \frac{\sin x}{2(1 - \cos x)}.$$

From this we can deduce that

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}$$

for n a positive-odd integer, where $\{B_n\}$ are the Bernoulli numbers.

There are numerous similar relations of the kind.

Another example is

$$\zeta\left(\tfrac{1}{2}(3 - \mathrm{i}\hat{h}_{\mathrm{BK}})\right) \sin x = \frac{\pi - x}{2}$$

for $x \in [0, \pi]$, from which we can deduce that $\zeta(s)$ vanishes for negative-even integers s without analytically continuing the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\tfrac{1}{2}\pi s\right) \Gamma(1-s)\zeta(1-s).$$

We can also deduce that $\zeta(0) = -\frac{1}{2}$, and that $\zeta(s)$ has a pole at $s = 1$.

1. Hamiltonian for the Riemann zeros

Consider the 'Hamiltonian' operator

$$\hat{H} = \frac{\mathbb{1}}{\mathbb{1} - \mathrm{e}^{-\mathrm{i}\hat{p}}} \left(\hat{x}\hat{p} + \hat{p}\hat{x}\right) \left(\mathbb{1} - \mathrm{e}^{-\mathrm{i}\hat{p}}\right)$$

on $\mathbb{R}^+ = (0, \infty)$.

With the boundary condition

$$\psi(0) = 0,$$

required to ensure

$$\langle \varphi, \hat{H}\psi \rangle = \langle \hat{H}\varphi, \psi \rangle,$$

the eigenvalues $\{E_n\}$ of $\hat{H}$ satisfy the property that $\left\{\frac{1}{2}(1 - \mathrm{i}E_n)\right\}$ are the nontrivial zeros of the Riemann zeta function.

The eigenstates of $\hat{H}$ are given by the Hurwitz zeta function

$$\psi_n(x) = -\zeta(z_n, x + 1),$$

with eigenvalues

$$E_n = \mathrm{i}(2z_n - 1).$$

2. The shift operator and its inverse

Defining

$$\hat{\Delta} \equiv \mathbb{1} - \mathrm{e}^{-\mathrm{i}\hat{p}},$$

in units $\hbar = 1$ we have

$$\hat{p} = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$$

so that

$$\hat{\Delta} f(x) = f(x) - f(x-1).$$

As for $\hat{\Delta}^{-1}$ we have

$$\hat{\Delta}^{-1} = \frac{\mathbb{1}}{\mathbb{1} - \mathrm{e}^{-\mathrm{i}\hat{p}}} = \frac{\mathbb{1}}{\mathrm{i}\hat{p}} \frac{-\mathrm{i}\hat{p}}{\mathrm{e}^{-\mathrm{i}\hat{p}} - \mathbb{1}} = \frac{\mathbb{1}}{\mathrm{i}\hat{p}} \sum_{n=0}^{\infty} B_n \frac{(-\mathrm{i}\hat{p})^n}{n!}.$$

In particular, if $f(x) \to 0$ sufficiently fast, then we have

$$\hat{\Delta}^{-1} f(x) = -\sum_{k=1}^{\infty} f(k+x).$$

3. Uniqueness of $\hat{\Delta}\psi$

We multiply the eigenvalue equation

$$\hat{H}\psi = E\psi$$

on the left by $\hat{\Delta}$.

Recall that

$$\hat{H} = \hat{\Delta}^{-1}(\hat{x}\hat{p} + \hat{p}\hat{x})\hat{\Delta}.$$

This gives a first-order linear differential equation

$$(\hat{x}\hat{p} + \hat{p}\hat{x})\hat{\Delta}\psi = -\mathrm{i}\left(2x\frac{\mathrm{d}}{\mathrm{d}x} + 1\right)\hat{\Delta}\psi = E\,\hat{\Delta}\psi$$

for the function $\hat{\Delta}\psi$, whose solution is unique and is given by

$$\hat{\Delta}\psi = x^{-z}$$

up to a multiplicative constant.

Therefore,

$$\psi(x) = \hat{\Delta}^{-1}x^{-z}.$$

4. Eigenstates and eigenvalues

To see $\hat{\Delta}^{-1}x^{-z} = -\zeta(z, x+1)$, observe that

$$\hat{\Delta}^{-1}x^{-z} = \frac{1}{\mathrm{i}\hat{p}}\sum_{n=0}^{\infty} B_n \frac{(-\mathrm{i}\hat{p})^n}{n!} x^{-z}$$
$$= \frac{1}{1-z}\sum_{n=0}^{\infty} B_n \frac{(-\mathrm{i}\hat{p})^n}{n!} x^{1-z}.$$

Because $\mathrm{i}\hat{p} = \partial_x$ and

$$\partial_x^n \, x^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)}\, x^{\mu-n},$$

setting $\mu = 1-z$ we find

$$\hat{\Delta}^{-1}x^{-z} = \frac{\Gamma(2-z)}{1-z}\sum_{n=0}^{\infty} B_n \frac{(-1)^n}{n!} \frac{x^{1-z-n}}{\Gamma(2-z-n)},$$

but we have $\Gamma(2-z) = (1-z)\Gamma(1-z)$ and

$$\frac{1}{\Gamma(2-z-n)} = \frac{1}{2\pi\mathrm{i}}\int_C \mathrm{d}u\, \mathrm{e}^u\, u^{n+z-2},$$

so

$$\begin{aligned}\hat{\Delta}^{-1}x^{-z} &= \frac{\Gamma(1-z)}{2\pi\mathrm{i}}\,x^{1-z}\int_C \mathrm{d}u\,\mathrm{e}^u\,u^{z-2}\sum_{n=0}^{\infty}B_n\frac{(-u/x)^n}{n!}\\ &= \frac{\Gamma(1-z)}{2\pi\mathrm{i}}\,x^{1-z}\int_C \mathrm{d}u\,\mathrm{e}^u u^{z-2}\frac{-u/x}{\mathrm{e}^{-u/x}-1}\end{aligned}$$

Now we scale the integration variable according to $u/x = t$ and obtain

$$\hat{\Delta}^{-1}x^{-z} = \frac{\Gamma(1-z)}{2\pi\mathrm{i}}\int_C \mathrm{d}t\,\frac{\mathrm{e}^{xt}\,t^{z-1}}{1-\mathrm{e}^{-t}} = -\zeta(z,x+1).$$

As for the eigenvalues, we have

$$\hat{H}\psi_z(x) = \hat{\Delta}^{-1}\left(\hat{x}\hat{p}+\hat{p}\hat{x}\right)\hat{\Delta}\hat{\Delta}^{-1}x^{-z} = \mathrm{i}(2z-1)\psi_z(x).$$

Because $\psi(0) = -\zeta(z,1) = -\zeta(z) = 0$, z is a zero of $\zeta(z)$.

5. Quantisation condition for the Berry-Keating Hamiltonian

Hamiltonian $\hat{H}$ is similar to the Berry-Keating Hamiltonian

$$\hat{h}_{\rm BK} = \hat{x}\,\hat{p} + \hat{p}\,\hat{x},$$

whose eigenstates and eigenvalues are

$$\phi_z^{\rm BK}(x) = x^{-z} \quad \text{and} \quad E = {\rm i}(2z-1).$$

The boundary condition $\psi(0) = 0$ then translates into the quantisation condition for the Berry-Keating Hamiltonian (as a boundary condition):

$$\varphi_z(0) = 0,$$

where

$$\varphi_z(x) := \phi_z^{\rm BK}(x) - \zeta(z,x).$$

Because on $\mathcal{L}^2(\mathbb{R}^+)$, $\hat{p}$ has a strictly positive imaginary part, both $\hat{\Delta}$ and $\hat{\Delta}^{-1}$ are bounded and invertible. $\Rightarrow$ $\hat{h}_{\rm BK}$ and $\hat{H}$ are isospectral.

But $\hat{h}_{\rm BK}$ is self-adjoint on $\mathbb{R}^+$, so that E is real.

6. Relations from Fourier analysis

Before we turn to the analysis of operator-valued zeta functions, we note the following relations (cf. Clausen 1832) in Fourier analysis

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^{2m}} = \frac{(-1)^{m-1}(2\pi)^{2m}}{2(2m)!} B_{2m}\left(\frac{x}{2\pi}\right)$$

and

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^{2m-1}} = \frac{(-1)^{m}(2\pi)^{2m-1}}{2(2m-1)!} B_{2m-1}\left(\frac{x}{2\pi}\right)$$

for $m = 1, 2, \ldots$, where $B_m(x)$ denotes the Bernoulli polynomial of order m.

If we set $m = 1$ in the sine series, we obtain

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \frac{\pi - x}{2}.$$

Series of this kind can be extended to cases for which m is 0 or a negative integer.

Such series can be summed by using Euler summation.

For example, we have

$$\sum_{n=1}^{\infty} \mathrm{e}^{\mathrm{i}nx} = \lim_{r\to 1^-} \sum_{n=1}^{\infty} \left(r\mathrm{e}^{\mathrm{i}x}\right)^n = \lim_{r\to 1^-} \frac{r\mathrm{e}^{\mathrm{i}x}}{1 - r\mathrm{e}^{\mathrm{i}x}} = \frac{1}{\mathrm{e}^{-\mathrm{i}x} - 1}.$$

Taking the imaginary part, we deduce that

$$\sum_{n=1}^{\infty} \sin(nx) = \frac{\sin x}{2(1 - \cos x)}.$$

7. The dilation generator and the Riemann dilation operator

Another ingredient we require is the notion of the dilation operator.

The generator of the dilation is $\hat{x}\hat{p}$, where $\hat{p} = -\mathrm{i}\,\mathrm{d}/\mathrm{d}x$, so

$$\mathrm{e}^{\mathrm{i}\lambda\hat{x}\hat{p}} f(x) = f(\mathrm{e}^{\lambda}x).$$

It follows that

$$\sin(nx) = n^{\mathrm{i}\hat{x}\hat{p}} \sin x.$$

Therefore, ignoring for now the question of the convergence of the sum, we find

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n} = \sum_{n=1}^{\infty} \frac{n^{\mathrm{i}\hat{x}\hat{p}}}{n} \sin x = \zeta\left(1 - \mathrm{i}\hat{x}\hat{p}\right) \sin x.$$

Thus, the action of the Riemann dilation operator $\zeta(1 - \mathrm{i}\hat{x}\hat{p})$ on a trigonometric function generates a Fourier series.

From $\hat{h}_{\mathrm{BK}} = 2\hat{x}\hat{p} - \mathrm{i}$ we have $1 - \mathrm{i}\hat{x}\hat{p} = \frac{1}{2}(3 - \mathrm{i}\hat{h}_{\mathrm{BK}})$.

It follows that

$$\zeta\left(1-\mathrm{i}\hat{x}\hat{p}\right)\sin x=\zeta\left(\tfrac{1}{2}(3-\mathrm{i}\hat{h}_{\mathrm{BK}})\right)\sin x=\frac{\pi-x}{2}.$$

Now if the operator $\zeta\left(1-\mathrm{i}\hat{x}\hat{p}\right)$ were invertible, then

$$\frac{1}{\zeta\left(1-\mathrm{i}\hat{x}\hat{p}\right)}\,\frac{\pi-x}{2}=\sin x.$$

However, since $\hat{x}\hat{p}$ is the dilation generator, it cannot change the power of x, so we arrive at a contradiction.

This suggests that the operator $\zeta\left(1-\mathrm{i}\hat{x}\hat{p}\right)$ *cannot be inverted* because its spectrum contains at least one zero eigenvalue.

8. Consistency check

To check the consistency without relying on the summation representation of $\zeta(s)$, we use the integral representation

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi\mathrm{i}} \int_C \frac{t^{s-1}}{\mathrm{e}^{-t}-1}\,\mathrm{d}t$$

for the zeta function and

$$\frac{1}{\Gamma(1-s)} = \frac{1}{2\pi\mathrm{i}} \int_C \mathrm{e}^t t^{s-1}\,\mathrm{d}t$$

for the reciprocal of the Gamma function.

Because s appears in two different ways, our strategy is to check the validity of

$$\frac{1}{\Gamma(\mathrm{i}\hat{x}\hat{p})} \frac{\pi - x}{2} = \frac{\zeta\left(1-\mathrm{i}\hat{x}\hat{p}\right)}{\Gamma(\mathrm{i}\hat{x}\hat{p})} \sin x.$$

For the left side we deduce that

$$\frac{1}{\Gamma(\mathrm{i}\hat{x}\hat{p})} \frac{\pi - x}{2} = \frac{1}{2\pi\mathrm{i}}\int_C \mathrm{e}^t t^{-\mathrm{i}\hat{x}\hat{p}} \left(\frac{\pi - x}{2}\right) \mathrm{d}t = \frac{1}{2\pi\mathrm{i}}\int_C \mathrm{e}^t \left(\frac{\pi - t^{-1}x}{2}\right) \mathrm{d}t = -\tfrac{1}{2}x.$$

The term $\pi/2$ has been annihilated because of the pole of $\zeta(s)$ at $s = 1$.

On the other hand, expanding $\sin x$ in a power series, we deduce from

$$\frac{\zeta\,(1-\mathrm{i}\hat{x}\hat{p})}{\Gamma(\mathrm{i}\hat{x}\hat{p})}\,x^n = \frac{1}{2\pi\mathrm{i}}\int_C \frac{t^{-\mathrm{i}\hat{x}\hat{p}}}{\mathrm{e}^{-t}-1}x^n\,\mathrm{d}t = \frac{1}{2\pi\mathrm{i}}\int_C \frac{t^{-n}}{\mathrm{e}^{-t}-1}x^n\,\mathrm{d}t = \frac{\zeta(1-n)}{\Gamma(n)}\,x^n$$

that

$$\frac{\zeta\,(1-\mathrm{i}\hat{x}\hat{p})}{\Gamma(\mathrm{i}\hat{x}\hat{p})}\,\sin x = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}\zeta(2(1-n))}{(2n-1)!\Gamma(2n-1)}\,x^{2n-1}.$$

Since the right side must equal $-\frac{1}{2}x$, we infer

$$\zeta(0) = -\frac{1}{2} \quad \text{and} \quad \zeta(-2) = \zeta(-4) = \cdots = 0.$$

Conversely, from these elementary facts about the zeta function we infer the consistency.

An essentially identical line of argument leads to the observation that

$$\zeta\left(2-\mathrm{i}\hat{x}\hat{p}\right)\,\cos x=\frac{\pi^2}{6}-\frac{\pi x}{2}+\frac{x^2}{4}$$

and that

$$\zeta\left(3-\mathrm{i}\hat{x}\hat{p}\right)\,\sin x=\frac{\pi^2 x}{6}-\frac{\pi x^2}{4}+\frac{x^3}{12},$$

and so on.

Thus, for each of the Clausen functions we obtain a corresponding representation in the form of an operator $\zeta\left(N-\mathrm{i}\hat{x}\hat{p}\right)$ acting on a trigonometric function, for N a positive integer.

Each of these relations reveals some information about the values of $\zeta(s)$ for real integral values of s.

9. Operator method and parity conservation

Because

$$\mathrm{i}\hat{x}\hat{p}\, x^{\alpha} = \alpha x^{\alpha}$$

and because $\zeta(s)$ is analytic except for a simple pole at $s = 1$, we have

$$\zeta(N - \mathrm{i}\hat{x}\hat{p})\, x^{n} = \zeta(N - n)\, x^{n}.$$

However, one must be careful about the existence of the pole.

To illustrate this, we consider the example $\zeta(1 - \mathrm{i}\hat{x}\hat{p})\,\sin x$.

Expanding the sine series, and assuming the interchangeability of the two limits, we obtain

$$\begin{aligned}\zeta(1 - \mathrm{i}\hat{x}\hat{p}) \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} &= \sum_{n=1}^{\infty} \zeta(1 - \mathrm{i}\hat{x}\hat{p}) \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} \\ &= \sum_{n=1}^{\infty} \zeta(2 - 2n) \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} = -\frac{1}{2} x.\end{aligned}$$

This shows that term-by-term application of the differential operator $\zeta(1 - \mathrm{i}\hat{x}\hat{p})$ is not permissible because we have missed the constant term $\pi/2$ associated with the pole of $\zeta(s)$.

In fact, for each of the examples discussed above, interchanging the limits leaves out just one term corresponding to the pole of $\zeta(s)$.

This term is the only parity-violating term.

10. Analyticity conservation

The term-by-term application of the operator respects both parity and analyticity.

To illustrate this, we consider the series

$$\sum_{n=1}^{\infty} \sin(nx) = \frac{\sin x}{2(1 - \cos x)}.$$

Each term on the left has odd parity and the right is also odd.

However, while each term on the left is analytic and vanishes at $x = 0$, the right side diverges like $1/x$ as $x \to 0$.

Indeed,

$$\sum_{n=0}^{\infty} \zeta(-\mathrm{i}\hat{x}\hat{p}) \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \zeta(-2n-1) \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$
$$= \frac{1}{x}\left[\sum_{k=0}^{\infty} \frac{1}{k!} B_k \, (\mathrm{i}x)^k - 1 - \mathrm{i}B_1 x\right].$$

Therefore, from the generating function $\sum_{k=0}^{\infty} B_k \, x^k/k! = x/(\mathrm{e}^x - 1)$ with $B_1 = -\frac{1}{2}$ we get

$$\sum_{n=0}^{\infty} \zeta(-\mathrm{i}\hat{x}\hat{p}) \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \frac{\sin x}{2(1-\cos x)} - \frac{1}{x}.$$

Remarkably, we recover the function, but with its singularity removed.

The singular term corresponds to the pole of $\zeta(s)$ at $s=1$.

Analogous results can be seen in other examples, for instance, in

$$\zeta\left(-1-\mathrm{i}\hat{x}\hat{p}\right)\,\cos x=\sum_{n=1}^{\infty}n\cos(nx)=-\frac{1}{2(1-\cos x)}.$$

Once again, there is no parity violation but the right side is singular at $x=0$ and behaves like $-1/x^2$, while each of the summands in the middle term is well behaved.

By interchanging the order of differentiation and summation we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\zeta\left(-1-\mathrm{i}\hat{x}\hat{p}\right)\frac{(-1)^n}{(2n)!}x^{2n} &= -\sum_{n=0}^{\infty}\frac{(2n+1)(\mathrm{i}x)^{2n}}{(2n+2)!}B_{2n+2}\\ &= -\frac{1}{2(1-\cos x)}+\frac{1}{x^2},\end{aligned}$$

and the singularity at the origin has been removed.

11. Discussion

We have only considered one class of operator-valued zeta functions, namely, zeta functions evaluated at a linear function of the dilation operator.

It appears that the action of this class of operators on trigonometric functions only yields information about $\zeta(s)$ for real s, but...

The matrix elements of, for example, $\zeta(1-\mathrm{i}\hat{x}\hat{p})$, in the standard sine basis $\{\sqrt{2/\pi}\,\sin(nx)\}$, is given by

$$\zeta_{mn} = \begin{cases} n/m & \text{if } n \text{ divides } m, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the matrix $\{\zeta_{mn}\}$ encodes the information about factorisation of integers.

This suggests that it might be possible to extract more information by studying further properties of the class of operator-valued zeta functions considered here.

To conclude, we have shown that by studying the action of Riemann dilation operators on trigonometric functions, we are able to infer some properties of the Riemann zeta function.

Of course, the properties of $\zeta(s)$ inferred here are already known.

Nevertheless, we were able to determine, for example, the locations of the trivial zeros from elementary Fourier analysis without relying explicitly on the analytic continuation of the zeta function.

This suggests that further research into actions of operator-valued zeta functions may yield interesting new results.

12. References

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