Eigenvector Correlations for the Ginibre Ensemble

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joint with Ron Rosenthal

The Ginibre Ensemble

Let $(m_{ij})_{i,j\in\mathbb{N}}$ be i.i.d. $\mathcal{N}_{\mathbb{C}}(0,1/N)$ variables. We consider the matrix

$$M_N := (m_{ij})_{i,j\leq N}$$

acting on $\mathbb{C}^N$.

- What are the statistical properties of this matrix ensemble?
- **Eigenvalues:** Almost surely, M_N is diagonalizable. With respect to Lebesgue measure $\prod_{i=1}^N \mathrm{d}^2\lambda_i$, the density is

$$\frac{\mathrm{d}\mathbb{P}(\underline{\lambda})}{\prod_{i=1}^N \mathrm{d}^2\lambda_i} = \frac{1}{Z_N}\prod_{i<j\leq N}|\lambda_i-\lambda_j|^2\prod_{i\leq N}\exp(-N|\lambda_i|^2)$$

- Asymptotic density of states is uniform over unit disc $\mathbf{D}_1 \subset \mathbb{C}$.

- Quite a bit is known about asymptotic behavior of eigenvalues in this ensemble and various generalizations.
  Ginibre '65; Girko '84, '94; Bai 97; Tao, Vu '08, '10; Götze, Tikhomirov '10; Bourgade, Yau, Yin '14a, '14b; Yin '14; Alt, Erdös, Krueger '18.
- Important fact: $\|([M_N - z]^*[M_N - z])^{-1/2}\|_\infty \sim N$, where as eigenvalue spacing is $N^{-1/2}$.
- In spite of this, much less is understood regarding the eigenvector geometry. Note however Rudelson, Vershynin '15.

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- In spite of this, much less is understood regarding the eigenvector geometry. Note however Rudelson, Vershynin '15.
- **Which Eigenvectors?** Given the eigenvalues $(\lambda_i)_{i=1}^N$, associate TWO bases:

$$\text{Column vectors:} \quad M_N \cdot r_i = \lambda_i r_i,$$
$$\text{Row vectors:} \quad \ell_i \cdot M_N = \lambda_i \ell_i,$$
$$\text{Normalization:} \quad \ell_i \cdot r_j = \delta_{i,j}.$$

- Then with $Q_i = r_i \otimes \ell_i$,

$$M_N = \sum_i \lambda_i Q_i.$$

Statistics of the Q_i's

Chalker-Mehlig '98:
Let $M_N(0), M_N(1)$ be independent copies of M_N and set

$$M_N(\theta) = \cos(\theta) M_N(0) + \sin(\theta) M_N(1).$$

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Then (at $\theta = 0$), eigenvalue trajectories $(\lambda_i(\theta))_{i \leq N}$ satisfy

$$\mathbb{E}[\partial_\theta \lambda_i \partial_\theta \overline{\lambda_j} | \lambda_i(0), \lambda_j(0)] = \frac{1}{N} \mathbb{E}[\mathrm{Tr}(Q_i^* Q_j) | \lambda_i(0), \lambda_j(0)],$$

$$\frac{1}{N} \mathbb{E}[\mathrm{Tr}(Q_i^* \cdot Q_j) | \lambda_i(0), \lambda_j(0)] \sim \begin{cases} 1 - |\lambda_i|^2 \text{ if } i = j, \\ -\frac{1}{N^2} \frac{1 - \overline{\lambda_i} \lambda_j}{|\lambda_i - \lambda_j|^4} \text{ if } i \neq j, \end{cases}$$

for typical eigenvalues.

Subsequent Work

- Burda, Nowak et al. ('99), Burda, Grela, Nowak et al. ('14), Belinshi, Nowak, et al. ('16), Nowak, Tarnowski ('18) ;
- Starr, Walters ('14). Corrections to **CM-'98** at $\partial \mathbf{D}_1$.
- Fyodorov ('17); Bourgade, Dubach ('18). Conditional on λ_i in bulk,

$$\frac{1}{N(1-|\lambda_i^2|)}\mathrm{Tr}[Q_i^* Q_i]$$

scales to $1/\Gamma(2)$.

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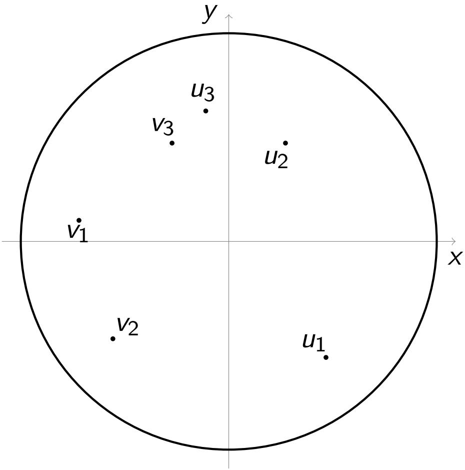

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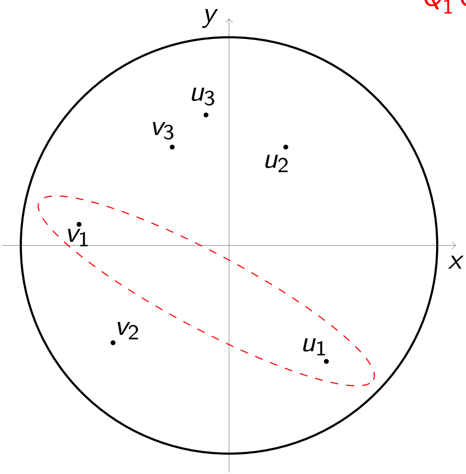

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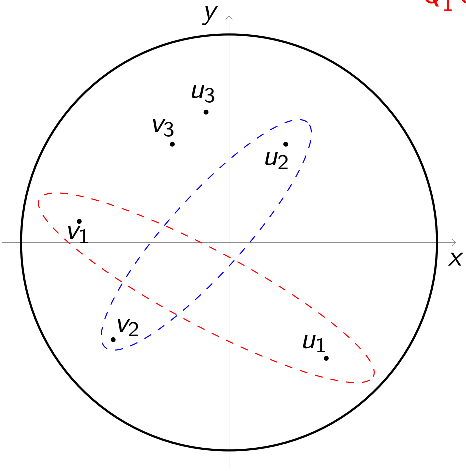

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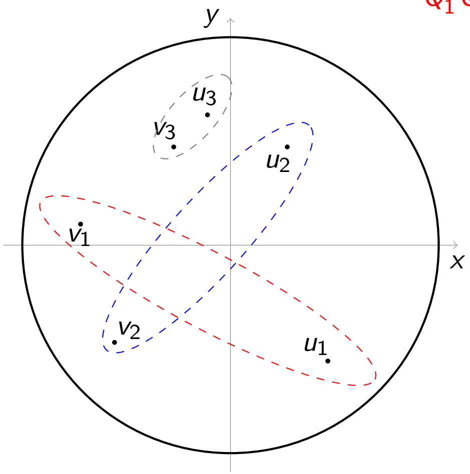

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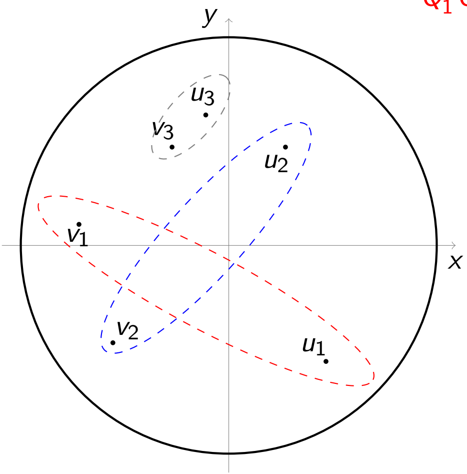

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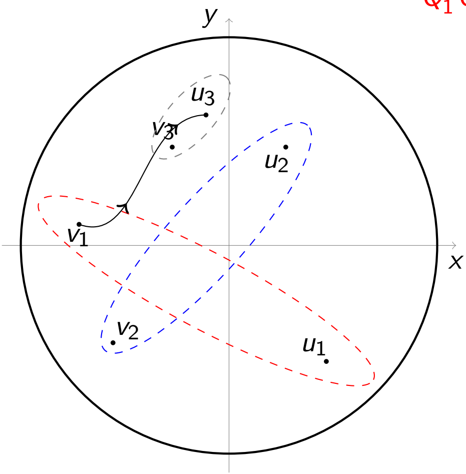

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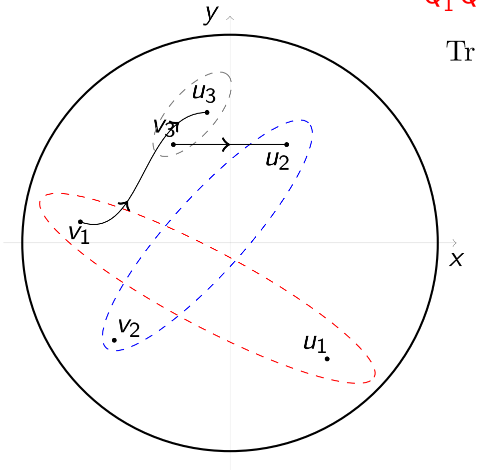

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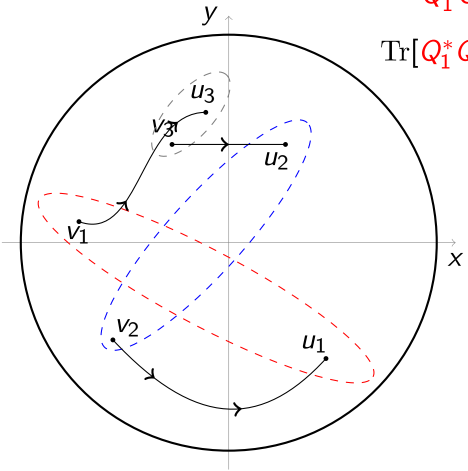

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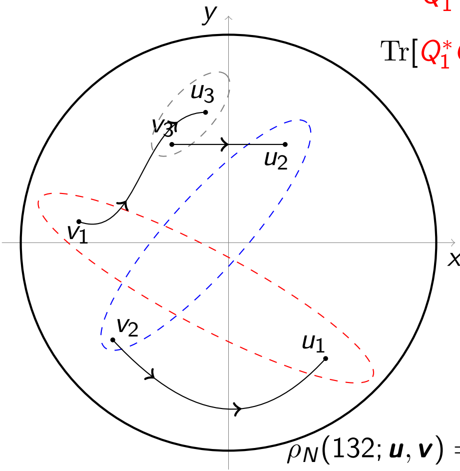

$$\rho_N(132; \boldsymbol{u}, \boldsymbol{v}) = \mathbb{E}[\hat{\rho}(132) | \lambda_{2j-1} = u_j, \lambda_{2j} = v_j]$$

- Given $k \in \mathbb{N}$ and $\mathbf{u}, \mathbf{v} \in \mathbf{D^k}$, we condition on

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- Let $\mathcal{S}_A$ be the permutation group on A.
- If $\mathcal{L} \in \mathcal{S}_A$ is a cycle on A let

$$\hat{\rho}(\mathcal{L}) = N^{|A|-1} \mathrm{Tr} \left[\prod_{j \in A} Q^*_{2j-1} Q_{2j} \right].$$

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  with cycle order imposed.
- For $\sigma \in \mathcal{S}_k$ set

$$\hat{\rho}(\sigma) = \prod_{\mathcal{L} \text{ cycles of } \sigma} \hat{\rho}(\mathcal{L}).$$

  Finally given $\mathbf{u}, \mathbf{v} \in \mathbf{D^k}$,

$$\rho_N(\sigma) = \mathbb{E}[\hat{\rho}(\sigma) | \lambda_{2j-1} = u_j, \lambda_{2j} = v_j \text{ for } j \in \{1, \dots k\}].$$

Computing $\lim_N \rho_N(\sigma)$, High Level

Let $R(z) = (M_N - z)^{-1}$

1. $Q_i = \oint_{|z-\lambda_i|=\varepsilon} \mathrm{d}z R(z)$.
   Corresponding $\hat{\rho}(\sigma)$, get $\bar{\rho}(\sigma)$ with (Q_{2i})'s replaced by $R(w_i)$'s (respectively Q_{2i+1}'s by $R(w_{2i+1})$'s).
2. Write $M_N = U_N T_N U_N^*$.
   Main Point: For Ginibre, can change of variables $M_N \to (U_N, T_N)$ so that

$$T_N = \begin{bmatrix} \lambda_1 & - & \vec{v}_1 & - & - \\ 0 & \lambda_2 & \vec{v}_2 & - & - \\ 0 & 0 & \lambda_3 & \vec{v}_3 & - \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \lambda_N \end{bmatrix}$$

and $\vec{v}_i \in \mathbb{C}^{N-i}$ are Gaussian with i.i.d. $\mathcal{N}_{\mathbb{C}}(0, 1/N)$ entries.

3. Define the $N-j \times N-j$ matrix

$$T^{(j)} = \begin{bmatrix} \lambda_j & - & \vec{v}_j & - & - \\ 0 & \lambda_{j+1} & \vec{v}_{j+1} & - & - \\ 0 & 0 & \lambda_{j+2} & \vec{v}_{j+2} & - \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \lambda_N \end{bmatrix}$$

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5. Viewing $\sigma \mapsto \bar{\rho}^{(j)}(\sigma)$ as a random vector, get

$$\mathbb{E}[\bar{\rho}^{(j)}(\sigma) | \underline{\lambda}, (\vec{v}_i)_{i \geq j}] = A_{\lambda_j} \cdot \bar{\rho}^{(j+1)}(\sigma)$$

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6. Hence

$$\mathbb{E}[\bar{\rho}(\sigma)|\underline{\lambda}] = A_{\lambda_1} \cdots A_{\lambda_N}(\mathrm{Id}, \sigma).$$

7. Study asymptotics.

- **Non Crossing Condition:** For $\sigma, \tau \in \mathcal{S}_k$ say $\sigma \preceq \tau$ if every cycle of σ is a subcycle of τ and $\tau \circ \sigma^{-1}$ has at most one nontrivial cycle. Let $\trianglelefteq$ be transitive closure of $\preceq$.

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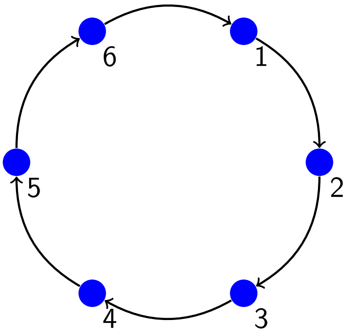

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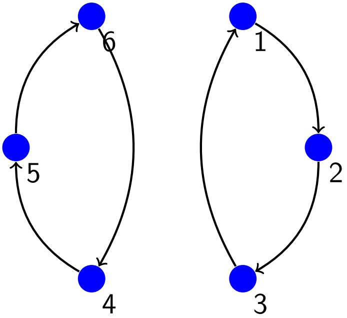

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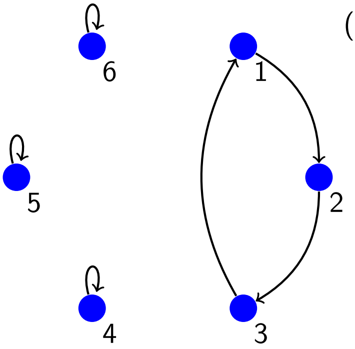

- Let

$$h(u,v) = \frac{1}{\pi}\int_{\mathbf{D}_1} \frac{1}{(\overline{\lambda}-\overline{u})(\lambda - v)}\mathrm{d}^2\lambda = \log\left(\frac{1-\overline{u}v}{|u-v|^2}\right).$$

and note that

$$\partial_u\partial_{\bar{v}} \exp(h(u,v)) = \partial_u\partial_{\bar{v}}\frac{1-\bar{u}v}{|u-v|^2} = -\frac{1-\bar{u}v}{|u-v|^4}$$

- This is **exactly** the correlation $\rho_2(u,v)$ Chalker and Mehlig computed.

Let

$$\mathfrak{h}_\sigma = \sum_{\alpha=1}^{k} h(u_\alpha, v_{\sigma^{-1}(\alpha)}).$$

$$\mathfrak{n}_{\sigma,\tau}(\mathbf{u},\mathbf{v}) = \frac{1}{\pi}\int_{\mathbf{D}_1} \prod_{\alpha:\tau\circ\sigma^{-1}\neq\alpha} \frac{1}{(\bar{\lambda}-\overline{u_\alpha})}\frac{1}{(\lambda - v_{\sigma^{-1}(\alpha)})}\mathrm{d}^2\lambda \qquad (1)$$

The matrix $\mathfrak{N} \equiv \mathfrak{N}(\mathbf{u},\mathbf{v})$ is then defined by

$$\mathfrak{N}(\sigma,\tau) = \begin{cases} \mathfrak{h}_\sigma(\mathbf{u},\mathbf{v}) & \text{if } \sigma=\tau, \\ \mathfrak{n}_{\sigma,\tau}(\mathbf{u},\mathbf{v}) & \text{if } \sigma \prec \tau, \\ 0 & \text{otherwise.} \end{cases} \qquad (2)$$

Note that

$$\mathfrak{N} = \sum_{\sigma\in\mathcal{S}_k} \mathfrak{h}_\sigma \mathfrak{q}_\sigma$$

Macroscopic Limit

For $\mathbf{u}, \mathbf{v} \in \mathbf{D}_1^k$, define

$$\mathrm{Dist}(\mathbf{u}, \mathbf{v}) := \min_{\alpha,\beta\in[k]} \{|u_\alpha - v_\beta|\} \wedge \min_{\alpha,\beta\in[k],\ \alpha\neq\beta} \{|u_\alpha - u_\beta|, |v_\alpha - v_\beta|\} \wedge \min_{\alpha\in[k]} \{1 - |u_\alpha|, 1 - |v_\alpha|\}. \quad (3)$$

Theorem

For every $\sigma \in \mathcal{S}_k$ and every $\mathbf{u}, \mathbf{v} \in \mathbf{D}_1^k$ such that $\mathrm{Dist}(\mathbf{u}, \mathbf{v}) > 0$, the limit

$$\rho(\sigma; \mathbf{u}, \mathbf{v}) := \lim_{N\to\infty} \rho_N(\sigma; \mathbf{u}, \mathbf{v})$$

exists.

Moreover

1. $\rho(\sigma) = \partial_{\mathbf{u}} \partial_{\overline{\mathbf{v}}} e^{\mathfrak{N}}(Id, \sigma)$,
2. *The $\mathfrak{q}_\sigma$ are rational in $\overline{\mathbf{u}}, \mathbf{v}$ (!).*
3. *$\rho(\sigma)$ factors over cycles.*

Correlation Structure of Cycle

- Using spectral decomposition

$$e^{\mathfrak{N}} = \sum_{\sigma \in \mathcal{S}_k} e^{\mathfrak{h}_\sigma} \mathfrak{q}_\sigma$$

and rationality of $\mathfrak{q}_\sigma$,

$$\partial_{\mathbf{u}} \partial_{\overline{\mathbf{v}}} e^{\mathfrak{N}} = \sum_{\sigma \in \mathcal{S}_k} [\partial_{\mathbf{u}} \partial_{\overline{\mathbf{v}}} e^{\mathfrak{h}_\sigma}] \mathfrak{q}_\sigma.$$

- $$[\partial_{\mathbf{u}} \partial_{\overline{\mathbf{v}}} e^{\mathfrak{h}_\sigma}] = \prod_{\alpha \in [k]} \rho_2(u_{\sigma(\alpha)}, v_\alpha)$$

Reminder:

$$\rho_2(z, w) = -\frac{1 - \overline{z} w}{|z - w|^4}$$

Corollary

There are two families of polynomials $(\mathfrak{R}_\sigma, \mathfrak{L}_\sigma)_{\sigma \in \mathcal{S}_\ell}$ *in* $\mathbf{u}, \mathbf{v} \in \mathbb{C}^k \times \mathbb{C}^k$, *homogeneous of degree of degree* $\binom{k-1}{2}$, *so that*

$$\rho(C_k; \mathbf{u}, \mathbf{v}) = \sum_{\sigma \trianglelefteq C_k} \frac{\mathfrak{R}_\sigma(\overline{\mathbf{u}}, \mathbf{v}) \mathfrak{L}_{C_k \circ \sigma^{-1}}(\overline{\mathbf{u}}, \sigma^{-1}(\mathbf{v}))}{\mathrm{V}_k(\overline{\mathbf{u}})^2 \mathrm{V}_k(\mathbf{v})^2} \prod_{\alpha \in [k]} \rho_2(u_{\sigma(\alpha)}, v_\alpha),$$

Example:

$$\rho_4(u_1, v_1, u_2, v_2) = \frac{1}{(\overline{u_1} - \overline{u_2})^2 (v_1 - v_2)^2} \Big[\rho_2(u_1, v_1) \rho_2(u_2, v_2) - \rho_2(u_1, v_2) \rho_2(u_2, v_1) \Big].$$