Eigenvectors of Non-Hermitian Random Matrices

Guillaume Dubach Courant Institute, NYU

October 8th, 2017 Random Matrices, Integrability and Complex Systems Yad Hashmona, Judean Hills, Israel

Joint work with Paul Bourgade

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Simulations

Ginibre Ensemble

Ginibre ensemble: $N \times N$ matrix $G = G_N$, with i.i.d. entries

$$G_{i,j} \stackrel{d}{=} \mathscr{N}\left(0, \frac{1}{N} \mathsf{Id}\right).$$

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Eigenvalues are almost surely distinct. We diagonalize

$$G = P\Delta P^{-1}, \quad \Delta = \text{Diag}(\lambda_1, \ldots, \lambda_N).$$

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Circular law: convergence of the empirical measure to the uniform measure on $\mathbb{D} = D(0, 1)$.

$$\sum_{k=1}^N \delta_{\lambda_k} \stackrel{d}{\to} \frac{1}{\pi} \mathbf{1}_{\mathbb{D}}.$$

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Circular Law

Ginibre, N=5000



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Overlaps of eigenvectors

 L_k : left eigenvector for λ_k . R_k : right eigenvector for λ_k .

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Matrix of overlaps:

$$\mathscr{O}_{ij} = \langle R_j \mid R_i \rangle \langle L_j \mid L_i \rangle$$

(Chalker & Mehlig '98, Walters & Starr '14).

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- In a sense, simplest homogeneous non trivial quantity.
- Quantify the **stability** of the spectrum. If $\lambda_i(t)$ is an eigenvalue of G + tE,

$$\mathscr{O}_{ii} = \lim_{t\to 0} \sup_{\|E\|=1} t^{-1} |\lambda_i(t) - \lambda_i|.$$

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• Appear naturally in **Ginibre Evolution**.

Simulations

Ginibre Evolution

Non-hermitian analog of Dyson Brownian Motion,

$$\mathrm{d}G_{ij}(t) = \frac{\mathrm{d}B_{ij}(t)}{\sqrt{N}} - \frac{1}{2}G_{ij}(t)\mathrm{d}t.$$

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$$\mathrm{d}G_{ij}(t) = \frac{\mathrm{d}B_{ij}(t)}{\sqrt{N}} - \frac{1}{2}G_{ij}(t)\mathrm{d}t.$$

Eigenvalues are correlated martingales without extra drift.

$$\mathrm{d}\lambda_k(t) = \mathrm{d}M_k(t) - \frac{1}{2}\lambda_k(t)\mathrm{d}t,$$

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with the bracket

$$\mathrm{d}\langle M_i,\overline{M_j}\rangle_t = \mathscr{O}_{i,j}(t) \frac{\mathrm{d}t}{N}$$

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Ginibre Evolution (Movie)

Main features : repulsion, slow 'speed' at the edge, surprising apparent correlation of some pairs or triplets of eigenvalues.

(Click to play video.)

Ginibre Evolution, N=700



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 Image: Image

First properties of overlaps

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Remark

For any *i*,
$$\mathcal{O}_{i,i} = ||R_i||^2 ||L_i||^2 \ge 1$$
 and $\sum_j \mathcal{O}_{i,j} = 1$.

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 and $\sum_j \mathcal{O}_{i,j} = 1$.

Proposition

The matrix \mathcal{O} is hermitian positive-definite with

$$\min \operatorname{Spec} \mathscr{O} = 1.$$

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Simulations

Diagonal Overlaps

Chalker & Mehlig computed the first moment of diagonal overlaps.

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Chalker & Mehlig computed the first moment of diagonal overlaps.

Proposition (Chalker and Mehlig)

Conditionally on
$$(\lambda_1, \dots, \lambda_N) = (z_1, \dots, z_N)$$
,
 $\mathbb{E}\left(\mathscr{O}_{11} | \lambda = \mathbf{z}\right) = \prod_{n=2}^N \left(1 + \frac{1}{N|z_1 - z_n|^2}\right)$,

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There is actually an explicit and simple decomposition of the quenched distribution of $\mathcal{O}_{1,1}$.

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Theorem (Bourgade, D.)

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where X_k 's are independent standard complex Gaussian.

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This enables to determine a limit distribution.

Theorem (Bourgade, D.)

Conditionally on $\lambda_1 = z_1 \in \mathbb{D}$,

$$N^{-1} \mathscr{O}_{1,1} \stackrel{d}{
ightarrow} (1 - |z_1|^2) \gamma_2^{-1}$$

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The γ_2^{-1} distribution



Figure: Density of $\frac{1}{\gamma_2}$, where γ_2 has density $\frac{1}{\Gamma(2)}te^{-t}\mathbf{1}_{\mathbb{R}_+}$.

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The γ_2^{-1} distribution



Figure: Density of $\frac{1}{\gamma_2}$, where γ_2 has density $\frac{1}{\Gamma(2)}te^{-t}\mathbf{1}_{\mathbb{R}_+}$.

Heavy-tail distribution (no second moment).

Off-diagonal overlaps

$$z_1, z_2 \in \mathbb{D}, \quad \omega = |z_1 - z_2| N^{1/2}.$$

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Off-diagonal overlaps

 $z_1, z_2 \in \mathbb{D}, \quad \omega = |z_1 - z_2| N^{1/2}.$ Mesoscopic scales : $\omega \sim N^{\epsilon}, \epsilon \in (0, \frac{1}{2}).$

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Theorem (Bourgade, D.)

Conditionally on $(\lambda_1, \lambda_2) = (z_1, z_2) \in \mathbb{D}^2$ at mesoscopic distance,

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Conditionally on $(\lambda_1, \lambda_2) = (z_1, z_2) \in \mathbb{D}^2$ at mesoscopic distance,

$$\begin{split} \mathbb{E}\left(\mathscr{O}_{12}\right) &\sim -\frac{1-z_1\overline{z_2}}{N|z_1-z_2|^4}\\ \mathbb{E}\left(|\mathscr{O}_{12}|^2\right) &\sim \frac{(1-|z_1|^2)^2}{|z_1-z_2|^4}\\ \mathbb{E}\left(\mathscr{O}_{11}\mathscr{O}_{22}\right) &\sim \mathbb{E}\left(\mathscr{O}_{11}\right)\mathbb{E}\left(\mathscr{O}_{22}\right) \end{split}$$

(First term was known by Chalker & Mehlig)

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Microscopic Scale

More importantly, one can go down to $\omega \sim 1$.

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Microscopic Scale

More importantly, one can go down to $\omega \sim 1$.

Theorem (Bourgade, D.)

Conditionally on $(\lambda_1,\lambda_2)=(z_1,z_2)\in \mathbb{D}^2$ at microscopic distance,

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More importantly, one can go down to $\omega \sim 1$.

Theorem (Bourgade, D.) Conditionally on $(\lambda_1, \lambda_2) = (z_1, z_2) \in \mathbb{D}^2$ at microscopic distance, $\mathbb{E}\left(\mathscr{O}_{12}\right) \sim -N\frac{1-z_1\overline{z_2}}{|\omega|^4} \times \frac{1-(1+|\omega|^2)e^{-|\omega|^2}}{1-e^{-|\omega|^2}}$ $\mathbb{E}\left(|\mathscr{O}_{12}|^2\right) \sim \frac{N^2(1-|z_1|^2)^2}{|_{(j)}|^4}$ $\mathbb{E}\left(\mathscr{O}_{11}\mathscr{O}_{22}\right) \sim \frac{N^2(1-|z_1|^2)^2}{|\omega|^4} \times \frac{1+|\omega|^4-e^{-|\omega|^2}}{1-e^{-|\omega|^2}}.$

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(First term conjectured by Chalker & Mehlig)

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Simulations

Sketch of proof

Theorem (Quenched distribution of the diagonal overlaps)

Conditionally on $(\lambda_1, \ldots, \lambda_N) = (z_1, \ldots, z_N) \in \mathbb{D}^N$,

$$\mathcal{O}_{11} \stackrel{\text{(d)}}{=} \prod_{k=2}^{N} \left(1 + \frac{|X_k|^2}{N|z_1 - z_k|^2} \right),$$

where X_k 's are independent standard complex Gaussian.

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To prove it, begin with Schur Decomposition :

 $G = UTU^*$

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Remark

T is independent on U.

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where X_k 's are independent standard complex Gaussian.

To prove it, begin with Schur Decomposition :

$$G = UTU^*$$

Remark

T is independent on U. The overlaps of the matrix T are the same as those of G !

Proofs

Simulations

Schur Decomposition :

 $G = UTU^*$

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Schur Decomposition :

 $G = UTU^*$

with

$$T = \begin{pmatrix} \lambda_1 & T_{12} & \dots & T_{1N} \\ 0 & \lambda_2 & \dots & T_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_N \end{pmatrix}.$$

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Proposition (Mehta)

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Proposition (Mehta)

The diagonal of T is independent of the upper-diagonal.

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Proposition (Mehta)

The diagonal of T is independent of the upper-diagonal. The upper-diagonal entries of T are i.i.d. $\mathcal{N}(0, \frac{1}{N})$.

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$$T = \begin{pmatrix} \lambda_1 & T_{12} & \dots & T_{1N} \\ 0 & \lambda_2 & \dots & T_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_N \end{pmatrix}$$

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$$T = \begin{pmatrix} \lambda_1 & T_{12} & \dots & T_{1N} \\ 0 & \lambda_2 & \dots & T_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_N \end{pmatrix}$$

Right-eigenvectors of $T: R_1 = (1, 0, ..., 0)$ $R_2 = (a, 1, 0, ..., 0).$

$$T = \begin{pmatrix} \lambda_1 & T_{12} & \dots & T_{1N} \\ 0 & \lambda_2 & \dots & T_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_N \end{pmatrix}$$

Right-eigenvectors of $T: R_1 = (1, 0, ..., 0) \quad R_2 = (a, 1, 0, ..., 0).$

Left-eigenvectors of T: $L_1 = (b_1, \ldots, b_N)$ $L_2 = (d_1, \ldots, d_N)$.

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$$T = \begin{pmatrix} \lambda_1 & T_{12} & \dots & T_{1N} \\ 0 & \lambda_2 & \dots & T_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_N \end{pmatrix}$$

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Left-eigenvectors of T: $L_1 = (b_1, \ldots, b_N)$ $L_2 = (d_1, \ldots, d_N)$.

with
$$a = -b_2, \ b_1 = 1, \quad b_i = rac{1}{\lambda_1 - \lambda_i} \sum_{k=1}^{i-1} b_k T_{ki} \quad ext{for } i \geq 2$$

and
$$d_1 = 0, \ d_2 = 1, \quad d_i = rac{1}{\lambda_2 - \lambda_i} \sum_{k=1}^{i-1} d_k T_{ki} \quad ext{for } i \geq 3.$$

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So, as $\mathscr{O}_{i,j} = \langle R_j | R_i \rangle \langle L_j | L_i \rangle$,

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$$\mathscr{O}_{11} = \sum_{i=1}^{N} |b_i|^2, \quad \mathscr{O}_{12} = -\overline{b_2} \sum_{i=2}^{N} b_i \overline{d_i}, \quad \mathscr{O}_{22} = (1+|b_2|^2) \sum_{i=2}^{N} |d_i|^2.$$

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$$\mathscr{O}_{11} = \sum_{i=1}^{N} |b_i|^2, \quad \mathscr{O}_{12} = -\overline{b_2} \sum_{i=2}^{N} b_i \overline{d_i}, \quad \mathscr{O}_{22} = (1+|b_2|^2) \sum_{i=2}^{N} |d_i|^2.$$

Define for $d \leq N$,

 $b^{(d)} = (b_1, \ldots, b_d)$

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$$\mathscr{O}_{11} = \sum_{i=1}^{N} |b_i|^2, \quad \mathscr{O}_{12} = -\overline{b_2} \sum_{i=2}^{N} b_i \overline{d_i}, \quad \mathscr{O}_{22} = (1+|b_2|^2) \sum_{i=2}^{N} |d_i|^2.$$

Define for $d \leq N$,

$$b^{(d)} = (b_1, \dots, b_d)$$
 $\mathscr{O}_{11}^{(d)} = \sum_{i=1}^d |b_i|^2 = \|b^{(d)}\|^2$

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$$\mathscr{O}_{11} = \sum_{i=1}^{N} |b_i|^2, \quad \mathscr{O}_{12} = -\overline{b_2} \sum_{i=2}^{N} b_i \overline{d_i}, \quad \mathscr{O}_{22} = (1+|b_2|^2) \sum_{i=2}^{N} |d_i|^2.$$

Define for $d \leq N$,

$$b^{(d)} = (b_1, \dots, b_d)$$
 $\mathscr{O}_{11}^{(d)} = \sum_{i=1}^d |b_i|^2 = \|b^{(d)}\|^2$
 $T_{d+1} = (T_{1,d+1}, T_{2,d+1}, \dots, T_{d,d+1})$

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$$\mathscr{O}_{11} = \sum_{i=1}^{N} |b_i|^2, \quad \mathscr{O}_{12} = -\overline{b_2} \sum_{i=2}^{N} b_i \overline{d_i}, \quad \mathscr{O}_{22} = (1+|b_2|^2) \sum_{i=2}^{N} |d_i|^2.$$

Define for $d \leq N$,

$$b^{(d)} = (b_1, \dots, b_d)$$

 $\mathscr{O}_{11}^{(d)} = \sum_{i=1}^d |b_i|^2 = \|b^{(d)}\|^2$
 $\mathcal{T}_{d+1} = (\mathcal{T}_{1,d+1}, \mathcal{T}_{2,d+1}, \dots, \mathcal{T}_{d,d+1})$

In this way,

$$b_{d+1} = \frac{1}{\lambda_1 - \lambda_{d+1}} b^{(d)} \cdot T_{d+1}.$$

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Initial and final terms :
$$\mathscr{O}_{1,1}^{(1)} = |b_1|^2 = 1$$
, $\mathscr{O}_{1,1}^{(N)} = \mathscr{O}_{1,1}$.

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Recurrence

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$$egin{aligned} \mathscr{O}_{1,1}^{(d+1)} &= \mathscr{O}_{1,1}^{(d)} + |b_{d+1}|^2 = \mathscr{O}_{1,1}^{(d)} + rac{1}{|\lambda_1 - \lambda_{d+1}|^2} |b^{(d)}. T_{d+1}| \ &= \mathscr{O}_{1,1}^{(d)} \left(1 + rac{1}{|\lambda_1 - \lambda_{d+1}|^2} rac{|b^{(d)}. T_{d+1}|^2}{\|b^{(d)}\|^2}
ight) \end{aligned}$$

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Recurrence

Initial and final terms :
$$\mathscr{O}_{1,1}^{(1)} = |b_1|^2 = 1, \quad \mathscr{O}_{1,1}^{(N)} = \mathscr{O}_{1,1}.$$

$$\begin{split} \mathscr{O}_{1,1}^{(d+1)} &= \mathscr{O}_{1,1}^{(d)} + |b_{d+1}|^2 = \mathscr{O}_{1,1}^{(d)} + \frac{1}{|\lambda_1 - \lambda_{d+1}|^2} |b^{(d)} \cdot T_{d+1}| \\ &= \mathscr{O}_{1,1}^{(d)} \left(1 + \frac{1}{|\lambda_1 - \lambda_{d+1}|^2} \frac{|b^{(d)} \cdot T_{d+1}|^2}{\|b^{(d)}\|^2} \right) \end{split}$$

Note that

$$X_{d+1} = \frac{\sqrt{N}b^{(d)} \cdot T_{d+1}}{\|b^{(d)}\|} \stackrel{d}{=} \mathcal{N}(0,1)$$

is independent from $\mathscr{O}_{1,1}^{(d)}.$ This yields the decomposition.

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Theorem (Limit distribution)

Conditioned on $\lambda_1 = z_1 \in \mathbb{D}$,

$$N^{-1} \mathscr{O}_{1,1} \to (1 - |z_1|^2) \gamma_2^{-1}$$

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Theorem (Limit distribution)

Conditioned on $\lambda_1 = z_1 \in \mathbb{D}$,

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Theorem (Kostlan I)

 $\{N|\lambda_1|^2, \ldots, N|\lambda_N|^2\}$ are distributed as independent $\{\gamma_1, \ldots, \gamma_N\}$ variables.

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Theorem (Limit distribution)

Conditioned on $\lambda_1 = z_1 \in \mathbb{D}$,

$$N^{-1} \mathscr{O}_{1,1} \to (1 - |z_1|^2) \gamma_2^{-1}$$

Theorem (Kostlan I)

 $\{N|\lambda_1|^2, \ldots, N|\lambda_N|^2\}$ are distributed as independent $\{\gamma_1, \ldots, \gamma_N\}$ variables.

Theorem (Kostlan II)

Conditioned on $\lambda_1 = 0$, $\{N|\lambda_2|^2, \ldots, N|\lambda_N|^2\}$ are distributed as independent $\{\gamma_2, \ldots, \gamma_N\}$ variables.

β - γ algebra

For a, b > 0 we recall the following facts. (\perp means independence.)

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β - γ algebra

For a, b > 0 we recall the following facts. (\perp means independence.)

Fact (1) If $\gamma_a \perp \gamma_b$, then $\frac{\gamma_a}{\gamma_a + \gamma_b} \stackrel{d}{=} \beta_{a,b}$.

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β - γ algebra

For a, b > 0 we recall the following facts. (\perp means independence.)

Fact (1) If $\gamma_a \perp \gamma_b$, then $\frac{\gamma_a}{\gamma_a + \gamma_b} \stackrel{d}{=} \beta_{a,b}$. Fact (2) If $\beta_{a,b} \perp \beta_{a+b,c}$, then $\beta_{a,b}\beta_{a+b,c} \stackrel{d}{=} \beta_{a,b+c}$.

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β - γ algebra

For a, b > 0 we recall the following facts. (\perp means independence.)

Fact (1) If $\gamma_{a} \perp \gamma_{b}$, then $\frac{\gamma_{a}}{\gamma_{a}+\gamma_{b}} \stackrel{d}{=} \beta_{a,b}$. Fact (2) If $\beta_{a,b} \perp \beta_{a+b,c}$, then $\beta_{a,b}\beta_{a+b,c} \stackrel{d}{=} \beta_{a,b+c}$.

Fact (3)
$$N\beta_{a,N} \xrightarrow[N \to \infty]{d} \gamma_a.$$

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Conditioned on $\lambda_1 = 0$, we can use the β - γ algebra.

$$\frac{1}{N} \mathscr{O}_{11} \quad \stackrel{\mathrm{(d)}}{=} \quad \frac{1}{N} \prod_{k=2}^{N} \left(1 + \frac{|X_k|^2}{N|\lambda_1 - \lambda_k|^2} \right)$$

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Proofs

Conditioned on $\lambda_1 = 0$, we can use the β - γ algebra.

$$\begin{split} \frac{1}{N} \mathscr{O}_{11} & \stackrel{\text{(d)}}{=} \quad \frac{1}{N} \prod_{k=2}^{N} \left(1 + \frac{|X_k|^2}{N|\lambda_1 - \lambda_k|^2} \right) \\ & = \quad \frac{1}{N} \prod_{k=2}^{N} \left(1 + \frac{|X_k|^2}{N|\lambda_k|^2} \right) \end{split}$$

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Proofs

Conditioned on $\lambda_1 = 0$, we can use the β - γ algebra.

$$\frac{1}{N} \mathscr{O}_{11} \stackrel{\text{(d)}}{=} \frac{1}{N} \prod_{k=2}^{N} \left(1 + \frac{|X_k|^2}{N|\lambda_1 - \lambda_k|^2} \right)$$
$$= \frac{1}{N} \prod_{k=2}^{N} \left(1 + \frac{|X_k|^2}{N|\lambda_k|^2} \right)$$
$$\stackrel{\text{(d)}}{=} \frac{1}{N} \prod_{k=2}^{N} \left(1 + \frac{\gamma_1}{\gamma_k} \right)$$

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Proofs

Conditioned on $\lambda_1 = 0$, we can use the β - γ algebra.

$$\begin{aligned} \frac{1}{N} \mathcal{O}_{11} &\stackrel{\text{(d)}}{=} & \frac{1}{N} \prod_{k=2}^{N} \left(1 + \frac{|X_k|^2}{N|\lambda_1 - \lambda_k|^2} \right) \\ &= & \frac{1}{N} \prod_{k=2}^{N} \left(1 + \frac{|X_k|^2}{N|\lambda_k|^2} \right) \\ &\stackrel{\text{(d)}}{=} & \frac{1}{N} \prod_{k=2}^{N} \left(1 + \frac{\gamma_1}{\gamma_k} \right) \\ &\stackrel{\text{(d)}}{=} & \frac{1}{N} \prod_{k=2}^{N} \beta_{k,1}^{-1} \end{aligned}$$

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Conditioned on $\lambda_1 = 0$, we can use the β - γ algebra.

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Conditioned on $\lambda_1 = 0$, we can use the β - γ algebra.

$$\begin{split} \frac{1}{N} \mathscr{O}_{11} & \stackrel{\text{(d)}}{=} & \frac{1}{N} \prod_{k=2}^{N} \left(1 + \frac{|X_k|^2}{N |\lambda_1 - \lambda_k|^2} \right) \\ &= & \frac{1}{N} \prod_{k=2}^{N} \left(1 + \frac{|X_k|^2}{N |\lambda_k|^2} \right) \\ &\stackrel{\text{(d)}}{=} & \frac{1}{N} \prod_{k=2}^{N} \left(1 + \frac{\gamma_1}{\gamma_k} \right) \\ &\stackrel{\text{(d)}}{=} & \frac{1}{N} \prod_{k=2}^{N} \beta_{k,1}^{-1} \\ &\stackrel{\text{(d)}}{=} & \frac{1}{N} \beta_{2,N-1}^{-1} \xrightarrow{d} \gamma_2^{-1}. \end{split}$$

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This is the limiting heavy-tail distribution that Chalker and Mehlig predicted.

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This is the limiting heavy-tail distribution that Chalker and Mehlig predicted.



Figure: Fact-checking over 100 Ginibre 600×600 matrices .

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Proofs

Simulations

How do we condition on $\lambda_1 = z_1$ anywhere in the bulk ?

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How do we condition on $\lambda_1 = z_1$ anywhere in the bulk ? Short-range vs long-range.



Figure: Domains of integration within the bulk

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Short-range vs long-range

Assume χ is smooth enough and has compact support.

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Short-range vs long-range

Assume χ is smooth enough and has compact support. Mesoscopic zoom $\theta = \theta(N) = N^{-1/2+\epsilon}$.

$$\chi_{\theta}(z) = \chi(z\theta^{-1})$$

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Short-range vs long-range

Assume χ is smooth enough and has compact support. Mesoscopic zoom $\theta = \theta(N) = N^{-1/2+\epsilon}$.

$$\chi_{\theta}(z) = \chi(z\theta^{-1})$$

$$\mathscr{O}_{11} \stackrel{(\mathrm{d})}{=} \prod_{n=2}^{N} \left(1 + \frac{|X_n|^2}{N|\lambda_1 - \lambda_n|^2} \right)$$

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Proofs

Short-range vs long-range

Assume χ is smooth enough and has compact support. Mesoscopic zoom $\theta = \theta(N) = N^{-1/2+\epsilon}$.

$$\chi_{\theta}(z) = \chi(z\theta^{-1})$$

$$\mathcal{O}_{11} \stackrel{\text{(d)}}{=} \prod_{n=2}^{N} \left(1 + \frac{|X_n|^2}{N|\lambda_1 - \lambda_n|^2} \right)$$
$$= e^{\left(\sum_{n=2}^{N} \log\left(1 + \frac{|X_n|^2}{N|\lambda_1 - \lambda_n|^2}\right)\chi_{\theta}(\lambda_n)\right)} \times e^{\left(\sum_{n=2}^{N} \log\left(1 + \frac{|X_n|^2}{N|\lambda_1 - \lambda_n|^2}\right)(1 - \chi_{\theta}(\lambda_n))\right)}$$

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Proofs

Short-range vs long-range

Assume χ is smooth enough and has compact support. Mesoscopic zoom $\theta = \theta(N) = N^{-1/2+\epsilon}$.

$$\chi_{\theta}(z) = \chi(z\theta^{-1})$$

$$\begin{aligned} \mathscr{O}_{11} &\stackrel{\text{(d)}}{=} &\prod_{n=2}^{N} \left(1 + \frac{|X_n|^2}{N|\lambda_1 - \lambda_n|^2} \right) \\ &= & e^{\left(\sum_{n=2}^{N} \log\left(1 + \frac{|X_n|^2}{N|\lambda_1 - \lambda_n|^2}\right)\chi_{\theta}(\lambda_n)\right)} \\ &\quad \times e^{\left(\sum_{n=2}^{N} \log\left(1 + \frac{|X_n|^2}{N|\lambda_1 - \lambda_n|^2}\right)(1 - \chi_{\theta}(\lambda_n))\right)} \\ &= & \mathscr{O}_{1,1}^{\text{short}} \mathscr{O}_{1,1}^{\log} \end{aligned}$$

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At any $\epsilon\text{-mesoscopic scale, i.e. }\theta=\textit{N}^{-1/2+\epsilon}\text{,}$

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At any ϵ -mesoscopic scale, i.e. $\theta = N^{-1/2+\epsilon}$,

• The short-range term doesn't depend on z_1 (invariance of local statistics).

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Proofs

At any ϵ -mesoscopic scale, i.e. $\theta = N^{-1/2+\epsilon}$,

• The short-range term doesn't depend on z_1 (invariance of local statistics). We compare it to the $z_1 = 0$ case and find

$$\mathscr{O}_{1,1}^{\mathsf{short}} \sim \mathit{N}^{2\epsilon} \gamma_2^{-1}.$$

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Proofs

At any ϵ -mesoscopic scale, i.e. $\theta = N^{-1/2+\epsilon}$,

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$$\mathscr{O}_{1,1}^{\mathsf{short}} \sim \mathit{N}^{2\epsilon} \gamma_2^{-1}.$$

• The long-range term is deterministic (rigidity).

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At any ϵ -mesoscopic scale, i.e. $\theta = N^{-1/2+\epsilon}$

 The short-range term doesn't depend on z₁ (invariance of local statistics). We compare it to the $z_1 = 0$ case and find

$$\mathscr{O}_{1,1}^{\mathrm{short}} \sim \mathit{N}^{2\epsilon} \gamma_2^{-1}.$$

• The long-range term is deterministic (rigidity). Compute an integral and

$$\mathscr{O}_{1,1}^{\mathsf{long}} \sim N^{1-2\epsilon} (1-|z_1|^2).$$

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At any ϵ -mesoscopic scale, i.e. $\theta = N^{-1/2+\epsilon}$,

• The short-range term doesn't depend on z_1 (invariance of local statistics). We compare it to the $z_1 = 0$ case and find

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$$\mathscr{O}_{1,1}^{\mathsf{long}} \sim N^{1-2\epsilon} (1-|z_1|^2).$$

$$\mathscr{O}_{1,1} = \mathscr{O}_{1,1}^{\mathsf{short}} \mathscr{O}_{1,1}^{\mathsf{long}}$$

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At any ϵ -mesoscopic scale, i.e. $\theta = N^{-1/2+\epsilon}$,

• The short-range term doesn't depend on z_1 (invariance of local statistics). We compare it to the $z_1 = 0$ case and find

$$\mathscr{O}_{1,1}^{\mathrm{short}} \sim \mathit{N}^{2\epsilon} \gamma_2^{-1}.$$

• The long-range term is deterministic (rigidity). Compute an integral and

$$\mathscr{O}_{1,1}^{\mathsf{long}} \sim N^{1-2\epsilon} (1-|z_1|^2).$$

$$\mathscr{O}_{1,1} = \mathscr{O}_{1,1}^{\mathrm{short}} \mathscr{O}_{1,1}^{\mathrm{long}} \sim N(1-|z_1|^2)\gamma_2^{-1}.$$

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At any ϵ -mesoscopic scale, i.e. $\theta = N^{-1/2+\epsilon}$,

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• The long-range term is deterministic (rigidity). Compute an integral and

$$\mathscr{O}_{1,1}^{\mathsf{long}} \sim N^{1-2\epsilon} (1-|z_1|^2).$$

$$\mathcal{O}_{1,1} = \mathcal{O}_{1,1}^{\mathsf{short}} \mathcal{O}_{1,1}^{\mathsf{long}} \sim \textit{N}(1 - |z_1|^2) \gamma_2^{-1}.$$

This gives the limit distribution of **diagonal overlaps** in the bulk.

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Off-diagonal overlaps

No limit distribution known, but explicit formulae for the first and second moments conditionally on $\lambda_1, \ldots, \lambda_N \in \mathbb{D}^N$.

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Proofs

Off-diagonal overlaps

No limit distribution known, but explicit formulae for the first and second moments conditionally on $\lambda_1, \ldots, \lambda_N \in \mathbb{D}^N$. We can integrate them, separating short-range from long-range terms.



Figure: Domains of integration for the off-diagonal overlaps

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Proofs

Simulations

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- 2. Results
- 3. Proofs
- 4. Simulations

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Simulations

Universality of the γ_2^{-1} limit (conjecture)



Figure: Histograms for i.i.d. non Gaussian entries.

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Eigenvectors of Non-Hermitian Random Matrices 30 / 32

Simulations

Universality of the γ_2^{-1} limit (conjecture)



Figure: Histograms for i.i.d. non Gaussian entries.

Complex Uniform. Complex Bernoulli -

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・ロト ・ 戸 ・ ・ ヨ ・ ・ Eigenvectors of Non-Hermitian Random Matrices 30 / 32

Ginibre Evolution : Color Movie

Consequence: average velocity of eigenvalues $\sim 1-|\lambda|^2,$ but the distribution has a heavy tail.

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Eigenvectors of Non-Hermitian Random Matrices 31 / 32

Ginibre Evolution : Color Movie

Consequence: average velocity of eigenvalues $\sim 1-|\lambda|^2,$ but the distribution has a heavy tail.

Colors are given according to the relative size of the associated diagonal overlaps : black, blue, magenta and red.

(Click to play video.)



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Eigenvectors of Non-Hermitian Random Matrices 31 / 32

Proofs

Simulations

References

Seminal articles by Chalker & Mehlig :

- Statistical properties of eigenvectors in non-Hermitian Gaussian random matrix ensembles.
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This presentation is based on **The distribution of overlaps between eigenvectors of Ginibre matrices**.

(Bourgade & D., 2018)

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