Eigenvectors of Non-Hermitian Random Matrices

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Random Matrices, Integrability and Complex Systems
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Joint work with Paul Bourgade
Contents

1. Definitions and motivations
2. Results
3. Proofs
4. Simulations
Ginibre Ensemble

Ginibre ensemble: $N \times N$ matrix $G = G_N$, with i.i.d. entries

$$G_{i,j} \overset{d}{=} \mathcal{N} \left( 0, \frac{1}{N} \text{Id} \right).$$
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Eigenvalues are almost surely distinct. We diagonalize

$$G = P \Delta P^{-1}, \quad \Delta = \text{Diag}(\lambda_1, \ldots, \lambda_N).$$
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$$G = P \Delta P^{-1}, \quad \Delta = \text{Diag}(\lambda_1, \ldots, \lambda_N).$$

Circular law: convergence of the empirical measure to the uniform measure on $\mathbb{D} = D(0, 1)$.

$$\sum_{k=1}^{N} \delta_{\lambda_k} \overset{d}{\to} \frac{1}{\pi} 1_{\mathbb{D}}.$$
Circular Law

Ginibre, N=5000
Overlaps of eigenvectors

$L_k$: left eigenvector for $\lambda_k$. \hspace{1cm} $R_k$: right eigenvector for $\lambda_k$. 

Matrix of overlaps:

$$O_{ij} = \langle R_j | R_i \rangle \langle L_j | L_i \rangle$$ (Chalker & Mehlig '98, Walters & Starr '14).

• In a sense, simplest homogeneous non-trivial quantity.
• Quantify the stability of the spectrum.

If $\lambda_i(t)$ is an eigenvalue of $G(t) + tE$,

$$O_{ii} = \lim_{t \to 0} \sup \|E\| = 1 \| \lambda_i(t) - \lambda_i \|.$$ 

• Appear naturally in Ginibre Evolution.
Overlaps of eigenvectors

$L_k$: left eigenvector for $\lambda_k$.  
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Chosen such that $\langle L_i \mid R_j \rangle = \delta_{i,j}$.
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Ginibre Evolution

Non-hermitian analog of Dyson Brownian Motion,

\[ \mathrm{d}G_{ij}(t) = \frac{\mathrm{d}B_{ij}(t)}{\sqrt{N}} - \frac{1}{2} G_{ij}(t) \mathrm{d}t. \]
Ginibre Evolution

Non-hermitian analog of Dyson Brownian Motion,

$$dG_{ij}(t) = \frac{dB_{ij}(t)}{\sqrt{N}} - \frac{1}{2} G_{ij}(t)dt.$$ 

Eigenvalues are \textbf{correlated martingales without extra drift}.

$$d\lambda_k(t) = dM_k(t) - \frac{1}{2} \lambda_k(t)dt,$$
Ginibre Evolution

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$$d\lambda_k(t) = dM_k(t) - \frac{1}{2} \lambda_k(t) dt,$$

with the bracket

$$d\langle M_i, M_j \rangle_t = \mathcal{O}_{i,j}(t) \frac{dt}{N}.$$
Ginibre Evolution (Movie)

Main features: repulsion, slow 'speed' at the edge, surprising apparent correlation of some pairs or triplets of eigenvalues.

(Click to play video.)
First properties of overlaps

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Remark

For any $i$, $O_{i,i} = \|R_i\|^2 \|L_i\|^2 \geq 1$ and $\sum_j O_{i,j} = 1$. 
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Proposition

The matrix $\mathcal{O}$ is hermitian positive-definite with

$$\min \text{Spec} \mathcal{O} = 1.$$
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Diagonal Overlaps

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**Proposition (Chalker and Mehlig)**

*Conditionally on* $(\lambda_1, \ldots, \lambda_N) = (z_1, \ldots, z_N)$,

$$
\mathbb{E}(\mathcal{O}_{11} \mid \lambda = \mathbf{z}) = \prod_{n=2}^{N} \left( 1 + \frac{1}{N |z_1 - z_n|^2} \right),
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**Proposition (Chalker and Mehlig)**

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\mathbb{E}(\mathcal{O}_{11}|\lambda = z) = \prod_{n=2}^{N} \left(1 + \frac{1}{N|z_1 - z_n|^2}\right),
\]

There is actually an explicit and simple decomposition of the quenched distribution of \(\mathcal{O}_{1,1}\).
Theorem (Bourgade, D.)

*Conditionally on* $(\lambda_1, \ldots, \lambda_N) = (z_1, \ldots, z_N)$,

\[
O_{11}(d) = N \prod_{k=2}^{N} \left(1 + \|X_k\|^2 \right)
\]

where \(X_k\)'s are independent standard complex Gaussian.

This enables to determine a limit distribution.
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*where* \(X_k\)'s *are independent standard complex Gaussian.*

This enables to determine a limit distribution.

**Theorem (Bourgade, D.)**

*Conditionally on* \(\lambda_1 = z_1 \in \mathbb{D}\),

\[
N^{-1} \mathcal{O}_{1,1} \xrightarrow{(d)} (1 - |z_1|^2) \gamma_2^{-1}
\]
The $\gamma_2^{-1}$ distribution

Figure: Density of $\frac{1}{\gamma_2}$, where $\gamma_2$ has density $\frac{1}{\Gamma(2)} te^{-t} 1_{\mathbb{R}^+_1}$. 
The $\gamma_2^{-1}$ distribution

Figure: Density of $\frac{1}{\gamma_2}$, where $\gamma_2$ has density $\frac{1}{\Gamma(2)} t e^{-t} 1_{\mathbb{R}^+}$.

Heavy-tail distribution (no second moment).
Off-diagonal overlaps

\[ z_1, z_2 \in \mathbb{D}, \quad \omega = |z_1 - z_2|N^{1/2}. \]
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Mesoscopic scales : \( \omega \sim N^\epsilon, \epsilon \in (0, \frac{1}{2}). \)
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**Theorem (Bourgade, D.)**

*Conditionally on \((\lambda_1, \lambda_2) = (z_1, z_2) \in \mathbb{D}^2 \) at mesoscopic distance,*

\[
E(O_{12}) \sim -1 - z_1 z_2 N^{1/4}
\]
\[
E(|O_{12}|^2) \sim (1 - |z_1|^2)^{1/2} |z_1 - z_2|^{1/4}
\]
\[
E(O_{11} O_{22}) \sim E(O_{11}) E(O_{22}).
\]

(First term was known by Chalker & Mehlig)
Off-diagonal overlaps

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\mathbb{E}(\mathcal{O}_{12}) \sim -\frac{1 - z_1 \overline{z_2}}{N |z_1 - z_2|^4}
\]

\[
\mathbb{E}(|\mathcal{O}_{12}|^2) \sim \frac{(1 - |z_1|^2)^2}{|z_1 - z_2|^4}
\]

\[
\mathbb{E}(\mathcal{O}_{11} \mathcal{O}_{22}) \sim \mathbb{E}(\mathcal{O}_{11}) \mathbb{E}(\mathcal{O}_{22}).
\]

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Microscopic Scale

More importantly, one can go down to $\omega \sim 1$. 
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\mathbb{E}(\theta_{12}) \sim -N \frac{1 - z_1 \overline{z_2}}{|\omega|^4} \times \frac{1 - (1 + |\omega|^2)e^{-|\omega|^2}}{1 - e^{-|\omega|^2}}
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\mathbb{E}(\theta_{12}^2) \sim \frac{N^2(1 - |z_1|^2)^2}{|\omega|^4}
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\mathbb{E}(\theta_{11} \theta_{22}) \sim \frac{N^2(1 - |z_1|^2)^2}{|\omega|^4} \times \frac{1 + |\omega|^4 - e^{-|\omega|^2}}{1 - e^{-|\omega|^2}}.
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$$\mathbb{E}(|\mathcal{O}_{12}|^2) \sim \frac{N^2(1 - |z_1|^2)^2}{|\omega|^4}$$

$$\mathbb{E}(\mathcal{O}_{11} \mathcal{O}_{22}) \sim \frac{N^2(1 - |z_1|^2)^2}{|\omega|^4} \times \frac{1 + |\omega|^4 - e^{-|\omega|^2}}{1 - e^{-|\omega|^2}}.$$  

(First term conjectured by Chalker & Mehlig)
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Sketch of proof

Theorem (Quenched distribution of the diagonal overlaps)

Conditionally on \((\lambda_1, \ldots, \lambda_N) = (z_1, \ldots, z_N) \in \mathbb{D}^N\),

\[
\mathcal{O}_{11} \overset{(d)}{=} \prod_{k=2}^{N} \left( 1 + \frac{|X_k|^2}{N|z_1 - z_k|^2} \right),
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where \(X_k\)'s are independent standard complex Gaussian.
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To prove it, begin with Schur Decomposition:

\[
G = U T U^* 
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Remark

\(T\) is independent on \(U\).
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To prove it, begin with Schur Decomposition:

$$G = UTU^*$$

Remark

$T$ is independent on $U$. The overlaps of the matrix $T$ are the same as those of $G$!
Schur Decomposition:

\[ G = U T U^* \]
Schur Decomposition:

\[ G = UTU^* \]

with

\[
T = \begin{pmatrix}
\lambda_1 & T_{12} & \cdots & T_{1N} \\
0 & \lambda_2 & \cdots & T_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \lambda_N
\end{pmatrix}.
\]
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Proposition (Mehta)
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**Proposition (Mehta)**

The diagonal of \( T \) is independent of the upper-diagonal.
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\end{pmatrix}.
\]

Proposition (Mehta)

The diagonal of \( T \) is independent of the upper-diagonal.

The upper-diagonal entries of \( T \) are i.i.d. \( \mathcal{N} \left( 0, \frac{1}{N} \right) \).
\[ T = \begin{pmatrix}
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Right-eigenvectors of T: \( R_1 = (1, 0, \ldots, 0) \quad R_2 = (a, 1, 0, \ldots, 0) \).
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Right-eigenvectors of $T$: $R_1 = (1, 0, \ldots, 0)$, $R_2 = (a, 1, 0, \ldots, 0)$.

Left-eigenvectors of $T$: $L_1 = (b_1, \ldots, b_N)$, $L_2 = (d_1, \ldots, d_N)$.
\[
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\end{pmatrix}
\]

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Left-eigenvectors of \( T \):
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with \( a = -b_2 \), \( b_1 = 1 \), \( b_i = \frac{1}{\lambda_1 - \lambda_i} \sum_{k=1}^{i-1} b_k T_{ki} \) for \( i \geq 2 \)

and \( d_1 = 0 \), \( d_2 = 1 \), \( d_i = \frac{1}{\lambda_2 - \lambda_i} \sum_{k=1}^{i-1} d_k T_{ki} \) for \( i \geq 3 \).
So, as $O_{i,j} = \langle R_j | R_i \rangle \langle L_j | L_i \rangle$,
So, as \( \mathcal{O}_{i,j} = \langle R_j \mid R_i \rangle \langle L_j \mid L_i \rangle \),

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\mathcal{O}_{11} = \sum_{i=1}^{N} |b_i|^2, \quad \mathcal{O}_{12} = -\overline{b_2} \sum_{i=2}^{N} b_i d_i, \quad \mathcal{O}_{22} = (1 + |b_2|^2) \sum_{i=2}^{N} |d_i|^2.
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Define for $d \leq N$,

$$b^{(d)} = (b_1, \ldots, b_d)$$
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Define for $d \leq N$,

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$$

In this way,

$$
b_{d+1} = \frac{1}{\lambda_1 - \lambda_{d+1}} b^{(d)} \cdot T_{d+1}.
$$
Recurrence

Initial and final terms: $\mathcal{O}_{1,1}^{(1)} = |b_1|^2 = 1$, $\mathcal{O}_{1,1}^{(N)} = \mathcal{O}_{1,1}$. 

Note that $X_{d+1} = \sqrt{N}b_{(d)}$. 

$\|b_{(d)}\|_2 = N(0,1)$ is independent from $\mathcal{O}_{1,1}^{(d)}$. This yields the decomposition. \[\square\]
Recurrence

Initial and final terms: $O_{1,1}^{(1)} = |b_1|^2 = 1$, $O_{1,1}^{(N)} = O_{1,1}^{(1)}$.

\[
O_{1,1}^{(d+1)} = O_{1,1}^{(d)} + |b_{d+1}|^2 = O_{1,1}^{(d)} + \frac{1}{|\lambda_1 - \lambda_{d+1}|^2} |b^{(d)} . T_{d+1}| \\
= O_{1,1}^{(d)} \left( 1 + \frac{1}{|\lambda_1 - \lambda_{d+1}|^2} \frac{|b^{(d)} . T_{d+1}|^2}{\|b^{(d)}\|^2} \right)
\]
Recurrence

Initial and final terms: \( \mathcal{O}_{1,1}^{(1)} = |b_1|^2 = 1, \quad \mathcal{O}_{1,1}^{(N)} = \mathcal{O}_{1,1} \).

\[
\mathcal{O}_{1,1}^{(d+1)} = \mathcal{O}_{1,1}^{(d)} + |b_{d+1}|^2 = \mathcal{O}_{1,1}^{(d)} + \frac{1}{|\lambda_1 - \lambda_{d+1}|^2} |b^{(d)} \cdot T_{d+1}| \\
= \mathcal{O}_{1,1}^{(d)} \left( 1 + \frac{1}{|\lambda_1 - \lambda_{d+1}|^2} \frac{|b^{(d)} \cdot T_{d+1}|^2}{\|b^{(d)}\|^2} \right)
\]

Note that

\[
X_{d+1} = \frac{\sqrt{N} b^{(d)} \cdot T_{d+1}}{\|b^{(d)}\|} \overset{d}{\sim} \mathcal{N}(0, 1)
\]

is independent from \( \mathcal{O}_{1,1}^{(d)} \). This yields the decomposition.
Theorem (Limit distribution)

Conditioned on $\lambda_1 = z_1 \in \mathbb{D}$,

$$N^{-1} \theta_{1,1} \to (1 - |z_1|^2) \gamma_2^{-1}$$
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### Theorem (Limit distribution)

**Conditioned on** \( \lambda_1 = z_1 \in \mathbb{D}, \)**

\[
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\]

### Theorem (Kostlan I)

\[\{N|\lambda_1|^2, \ldots, N|\lambda_N|^2\}\] are distributed as independent \( \{\gamma_1, \ldots, \gamma_N\} \) variables.

### Theorem (Kostlan II)

**Conditioned on** \( \lambda_1 = 0, \) \( \{N|\lambda_2|^2, \ldots, N|\lambda_N|^2\}\) are distributed as independent \( \{\gamma_2, \ldots, \gamma_N\} \) variables.
For $a, b > 0$ we recall the following facts. ($\perp$ means independence.)

**Fact (1)**
If $\gamma_a \perp \gamma_b$, then $\gamma_a \gamma_a + \gamma_b = \beta_{a,b}$.

**Fact (2)**
If $\beta_{a,b} \perp \beta_{a,b} + b, c$, then $\beta_{a,b} \beta_{a,b} + b, c = \beta_{a,b} + b, c$.

**Fact (3)**
$N \beta_{a,b} \rightarrow N \gamma_a$ as $N \rightarrow \infty$. 
For $a, b > 0$ we recall the following facts. ($\perp$ means independence.)

**Fact (1)**

If $\gamma_a \perp \gamma_b$, then $\frac{\gamma_a}{\gamma_a + \gamma_b} \overset{d}{=} \beta_{a,b}$. 

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If \( \beta_{a,b} \perp \beta_{a+b,c} \), then \( \beta_{a,b} \beta_{a+b,c} \overset{d}{=} \beta_{a,b+c} \).
**β-γ algebra**

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**Fact (3)**

$\mathcal{N} \beta_{a,N} \overset{d}{\to} \gamma_a$.
Conditioned on $\lambda_1 = 0$, we can use the $\beta-\gamma$ algebra.

$$\frac{1}{N} \mathcal{O}_{11} \overset{(d)}{=} \frac{1}{N} \prod_{k=2}^{N} \left( 1 + \frac{|X_k|^2}{N|\lambda_1 - \lambda_k|^2} \right)$$
Conditioned on $\lambda_1 = 0$, we can use the $\beta$-$\gamma$ algebra.

\[
\frac{1}{N} \Theta_{11} \overset{(d)}{=} \frac{1}{N} \prod_{k=2}^{N} \left( 1 + \frac{|X_k|^2}{N|\lambda_1 - \lambda_k|^2} \right)
\]

\[
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\]
Conditioned on $\lambda_1 = 0$, we can use the $\beta$-$\gamma$ algebra.

\[
\frac{1}{N} \mathcal{O}_{11} \overset{(d)}{=} \frac{1}{N} \prod_{k=2}^{N} \left( 1 + \frac{|X_k|^2}{N|\lambda_1 - \lambda_k|^2} \right)
\]

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\]

\[
\overset{(d)}{=} \frac{1}{N} \beta_{2,N-1}^{-1} \overset{d}{\longrightarrow} \frac{d}{N \to \infty} \gamma_2^{-1}.
\]
\[ \frac{O_{11}}{N} \xrightarrow{d_{\rightarrow \infty}} \gamma_{2}^{-1} \]

This is the limiting heavy-tail distribution that Chalker and Mehlig predicted.
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\[
\frac{O_{11}}{N} \xrightarrow{d \, N \to \infty} \gamma_2^{-1}
\]

**Figure:** Fact-checking over 100 Ginibre $600 \times 600$ matrices.
How do we condition on $\lambda_1 = z_1$ anywhere in the bulk?
How do we condition on $\lambda_1 = z_1$ anywhere in the bulk?

Short-range vs long-range.

**Figure:** Domains of integration within the bulk
Short-range vs long-range

Assume $\chi$ is smooth enough and has compact support.
Short-range vs long-range

Assume $\chi$ is smooth enough and has compact support. Mesoscopic zoom $\theta = \theta(N) = N^{-1/2+\epsilon}$.

$$\chi_\theta(z) = \chi(z^{\theta^{-1}})$$
Assume $\chi$ is smooth enough and has compact support. Mesoscopic zoom $\theta = \theta(N) = N^{-1/2+\epsilon}$.

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$$\mathcal{O}_{11} \overset{(d)}{=} \prod_{n=2}^{N} \left(1 + \frac{|X_n|^2}{N|\lambda_1 - \lambda_n|^2}\right)$$
Assume $\chi$ is smooth enough and has compact support. Mesoscopic zoom $\theta = \theta(N) = N^{-1/2+\epsilon}$.

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$$\mathcal{O}_{11} \overset{(d)}{=} \prod_{n=2}^{N} \left( 1 + \frac{|X_n|^2}{N|\lambda_1 - \lambda_n|^2} \right)$$

$$= e^{\left( \sum_{n=2}^{N} \log \left( 1 + \frac{|X_n|^2}{N|\lambda_1 - \lambda_n|^2} \right) \chi_\theta(\lambda_n) \right)} \times e^{\left( \sum_{n=2}^{N} \log \left( 1 + \frac{|X_n|^2}{N|\lambda_1 - \lambda_n|^2} \right) (1 - \chi_\theta(\lambda_n)) \right)}$$
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$$= \Theta_{1,1}^{\text{short}} \Theta_{1,1}^{\text{long}}$$
At any $\epsilon$-mesoscopic scale, i.e. $\theta = N^{-1/2+\epsilon}$,
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- The short-range term doesn’t depend on $z_1$ (invariance of local statistics).
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- The short-range term doesn’t depend on $z_1$ (invariance of local statistics). We compare it to the $z_1 = 0$ case and find

$$\mathcal{O}_{1,1}^{\text{short}} \sim N^{2\epsilon} \gamma_2^{-1}.$$
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This gives the limit distribution of diagonal overlaps in the bulk.
Off-diagonal overlaps

No limit distribution known, but explicit formulae for the first and second moments conditionally on $\lambda_1, \ldots, \lambda_N \in \mathbb{D}^N$. 
Off-diagonal overlaps

No limit distribution known, but explicit formulae for the first and second moments conditionally on $\lambda_1, \ldots, \lambda_N \in \mathbb{D}^N$. We can integrate them, separating short-range from long-range terms.

**Figure:** Domains of integration for the off-diagonal overlaps
Contents

1. Definitions and motivations
2. Results
3. Proofs
4. Simulations
Universality of the $\gamma_2^{-1}$ limit (conjecture)

Figure: Histograms for i.i.d. non Gaussian entries.
Universality of the $\gamma_2^{-1}$ limit (conjecture)

Figure: Histograms for i.i.d. non Gaussian entries.

Complex Bernoulli  –  Complex Uniform.
Ginibre Evolution : Color Movie

Consequence: average velocity of eigenvalues $\sim 1 - |\lambda|^2$, but the distribution has a heavy tail.
**Ginibre Evolution: Color Movie**

Consequence: average velocity of eigenvalues $\sim 1 - |\lambda|^2$, but the distribution has a heavy tail.

Colors are given according to the relative size of the associated diagonal overlaps: black, blue, magenta and red.

(Click to play video.)
Seminal articles by Chalker & Mehlig:

- **Statistical properties of eigenvectors in non-Hermitian Gaussian random matrix ensembles.**
- **Eigenvector statistics in non-Hermitian random matrix ensembles.**

Recent related works:
- Fyodorov (2017),
- Crawford & Rosenthal (2018),
- Nowak & Tarnowski (2018),

This presentation is based on
- The distribution of overlaps between eigenvectors of Ginibre matrices.
  (Bourgade & D., 2018)
References

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