Eigenvectors of Non-Hermitian Random Matrices

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Random Matrices, Integrability and Complex Systems
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Joint work with Paul Bourgade

Contents

Ginibre Ensemble

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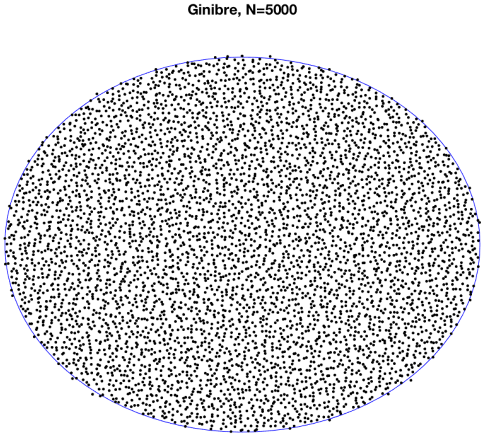
Circular law: convergence of the empirical measure to the uniform measure on $\mathbb{D} = D(0,1)$.

$$\sum_{k=1}^{N} \delta_{\lambda_k} \xrightarrow{d} \frac{1}{\pi} \mathbf{1}_{\mathbb{D}}.$$

Circular Law



Overlaps of eigenvectors

L_k: left eigenvector for λ_k. R_k: right eigenvector for λ_k.

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$$\mathscr{O}_{ij} = \langle R_j \mid R_i \rangle \langle L_j \mid L_i \rangle$$

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- Quantify the **stability** of the spectrum.
  If $\lambda_i(t)$ is an eigenvalue of $G + tE$,

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- Appear naturally in **Ginibre Evolution**.

Ginibre Evolution

Non-hermitian analog of Dyson Brownian Motion,

$$\mathrm{d}G_{ij}(t) = \frac{\mathrm{d}B_{ij}(t)}{\sqrt{N}} - \frac{1}{2}G_{ij}(t)\mathrm{d}t.$$

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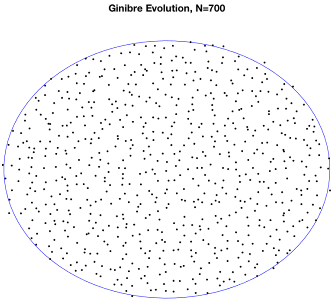
with the bracket

$$\mathrm{d}\langle M_i, \overline{M_j}\rangle_t = \mathscr{O}_{i,j}(t)\frac{\mathrm{d}t}{N}.$$

Ginibre Evolution (Movie)

Main features : repulsion, slow 'speed' at the edge, surprising apparent correlation of some pairs or triplets of eigenvalues.

(Click to play video.)



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Proposition

The matrix $\mathscr{O}$ is hermitian positive-definite with

$$\min \mathrm{Spec} \mathscr{O} = 1.$$

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Proposition (Chalker and Mehlig)

Conditionally on $(\lambda_1, \dots, \lambda_N) = (z_1, \dots, z_N)$,

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There is actually an explicit and simple decomposition of the quenched distribution of $\mathscr{O}_{1,1}$.

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Conditionally on $\lambda_1 = z_1 \in \mathbb{D}$,

$$N^{-1}\mathscr{O}_{1,1} \overset{d}{\to} (1 - |z_1|^2)\gamma_2^{-1}$$

The γ_2^{-1} distribution

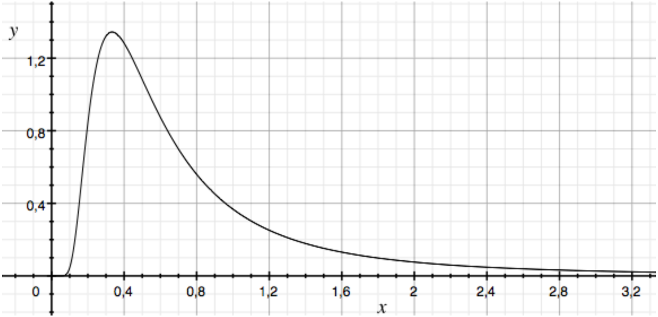

Figure: Density of $\frac{1}{\gamma_2}$, where γ_2 has density $\frac{1}{\Gamma(2)} t e^{-t} \mathbf{1}_{\mathbb{R}_+}$.

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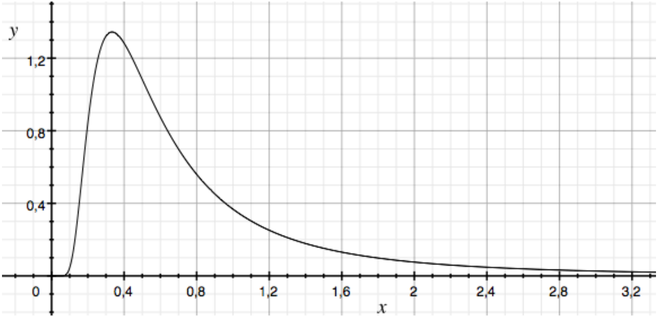

Figure: Density of $\frac{1}{\gamma_2}$, where γ_2 has density $\frac{1}{\Gamma(2)} te^{-t}\mathbf{1}_{\mathbb{R}_+}$.

Heavy-tail distribution (no second moment).

Off-diagonal overlaps

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$$\mathbb{E}\left(\mathscr{O}_{12}\right) \sim -\frac{1 - z_1 \overline{z_2}}{N|z_1 - z_2|^4}$$

$$\mathbb{E}\left(|\mathscr{O}_{12}|^2\right) \sim \frac{(1 - |z_1|^2)^2}{|z_1 - z_2|^4}$$

$$\mathbb{E}\left(\mathscr{O}_{11}\mathscr{O}_{22}\right) \sim \mathbb{E}\left(\mathscr{O}_{11}\right)\mathbb{E}\left(\mathscr{O}_{22}\right).$$

(First term was known by Chalker & Mehlig)

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Theorem (Quenched distribution of the diagonal overlaps)

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The overlaps of the matrix T are the same as those of G !

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The upper-diagonal entries of T are i.i.d. $\mathscr{N}\left(0, \frac{1}{N}\right)$.

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$$\text{with } a = -b_2, \; b_1 = 1, \quad b_i = \frac{1}{\lambda_1 - \lambda_i} \sum_{k=1}^{i-1} b_k T_{ki} \quad \text{for } i \geq 2$$

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In this way,

$$b_{d+1} = \frac{1}{\lambda_1 - \lambda_{d+1}} b^{(d)} . T_{d+1}.$$

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Note that

$$X_{d+1} = \frac{\sqrt{N} b^{(d)}.T_{d+1}}{\|b^{(d)}\|} \overset{d}{=} \mathscr{N}(0,1)$$

is independent from $\mathscr{O}_{1,1}^{(d)}$. This yields the decomposition.

□

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Theorem (Kostlan I)

$\{N|\lambda_1|^2, \ldots, N|\lambda_N|^2\}$ *are distributed as independent* $\{\gamma_1, \ldots, \gamma_N\}$ *variables.*

Theorem (Limit distribution)

Conditioned on $\lambda_1 = z_1 \in \mathbb{D}$,

$$N^{-1}\mathscr{O}_{1,1} \to (1 - |z_1|^2)\gamma_2^{-1}$$

Theorem (Kostlan I)

$\{N|\lambda_1|^2, \dots, N|\lambda_N|^2\}$ *are distributed as independent* $\{\gamma_1, \dots, \gamma_N\}$ *variables.*

Theorem (Kostlan II)

Conditioned on $\lambda_1 = 0$, $\{N|\lambda_2|^2, \dots, N|\lambda_N|^2\}$ *are distributed as independent* $\{\gamma_2, \dots, \gamma_N\}$ *variables.*

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For $a, b > 0$ we recall the following facts. ($\perp$ means independence.)

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Fact (3)

$N\beta_{a,N} \xrightarrow[N\to\infty]{d} \gamma_a$.

Conditioned on $\lambda_1 = 0$, we can use the β-γ algebra.

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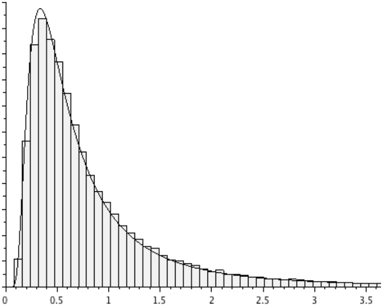

Figure: Fact-checking over 100 Ginibre 600×600 matrices .

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Short-range vs long-range.

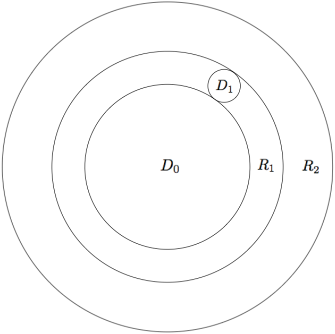

Figure: Domains of integration within the bulk

Short-range vs long-range

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&= \mathscr{O}_{1,1}^{\text{short}}\mathscr{O}_{1,1}^{\text{long}}
\end{aligned}$$

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This gives the limit distribution of **diagonal overlaps** in the bulk.

Off-diagonal overlaps

No limit distribution known, but explicit formulae for the first and second moments conditionally on $\lambda_1, \ldots, \lambda_N \in \mathbb{D}^N$.

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No limit distribution known, but explicit formulae for the first and second moments conditionally on $\lambda_1, \ldots, \lambda_N \in \mathbb{D}^N$. We can integrate them, separating short-range from long-range terms.

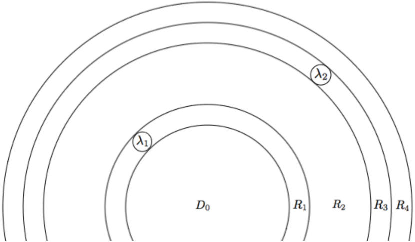

Figure: Domains of integration for the off-diagonal overlaps

Contents

Universality of the γ_2^{-1} limit (conjecture)

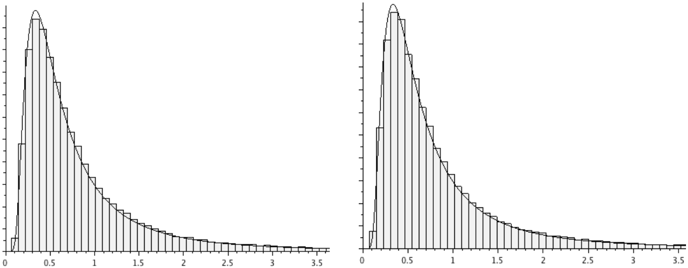

Figure: Histograms for i.i.d. non Gaussian entries.

Universality of the γ_2^{-1} limit (conjecture)

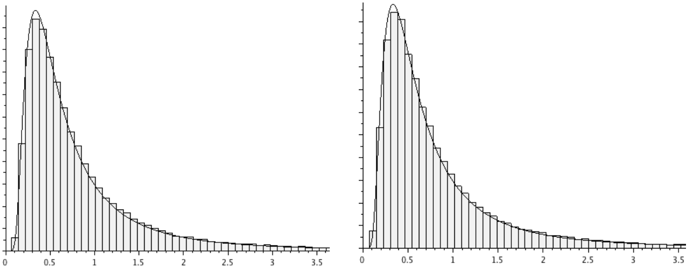

Figure: Histograms for i.i.d. non Gaussian entries.

Complex Bernoulli – Complex Uniform.

Ginibre Evolution : Color Movie

Consequence: average velocity of eigenvalues $\sim 1 - |\lambda|^2$, but the distribution has a heavy tail.

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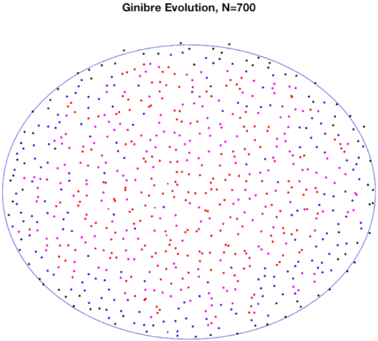
Consequence: average velocity of eigenvalues $\sim\ 1-|\lambda|^2$, but the distribution has a heavy tail.

Colors are given according to the relative size of the associated diagonal overlaps : black, blue, magenta and red.

(Click to play video.)



References

Seminal articles by Chalker & Mehlig :

- **Statistical properties of eigenvectors in non-Hermitian Gaussian random matrix ensembles**.
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This presentation is based on **The distribution of overlaps between eigenvectors of Ginibre matrices**.

(Bourgade & D., 2018)