Bound on random matrix theory to describe local observables of many-body systems

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Random Matrices, Integrability and Complex Systems

October 7, 2018

Motivation

- Thermalization (dynamics) of many-body quantum systems
- Connection between Eigenstate Themalization and dynamics
- Role of Random Matrix Theory and relevant (time)scales

Thermalization of Quantum Systems

Eigenstate Thermalization Hypothesis

- Individual energy eigenstate is "thermal"

$$\langle E_i|A|E_i\rangle \simeq \mathrm{Tr}(\rho_{\mathrm{mic}} A) \simeq \mathrm{Tr}(e^{-\beta H} A)/\mathrm{Tr}(e^{-\beta H})$$

- "Eigenstate Ensemble" explains eventual thermalization

$$\lim_{t\to\infty} \langle \Psi(t)|A|\Psi(t)\rangle = \sum_i |C_i|^2 \langle E_i|A|E_i\rangle +$$

$$\lim_{t\to\infty} \sum_{i\neq j} C_i^* C_j \langle E_i|A|E_j\rangle e^{-i(E_i-E_j)t} \simeq A^{\mathrm{th}} + O(1/L)$$

Eigenstate Thermalization Hypothesis

- ETH ansatz

$$\langle E_i|A|E_j\rangle = A^{\rm eth}(E)\delta_{ij} + \Omega^{-1/2} f(E,\omega) r_{ij}$$

· $E = (E_i + E_j)/2,\ \omega = E_i - E_j$
· $A^{\rm eth}, f$ depend on energy density E/V

Deutsch'91   Srednicki'94; 99   Rigol, Dunjko, Olshanii'08

- Meaning of form-factor $f(\omega)$:

$$\langle A(t)A(0)\rangle_\beta = \int d\omega\, f^2(E,\omega)\, e^{-i\omega t}$$

Chaoticity, ETH and Random Matrices

- Chaotic behavior: Hamiltonian = Random Matrix (WD distribution of energy levels)
- ETH $\simeq$ Eigenstates are random vectors
- "random" behavior of r_{ij}, i.e. A_{ij} with $i \neq j$ (empirical evidence)
- universal "ergodic" behavior of observables $\langle \Psi | A(t) | \Psi \rangle$ for large t (after thermalization) $\Leftrightarrow$ "structureless" or Haar-invariant A_{ij}
  D'Alessio, Kafri, Polkovnikov, Rigol'15
  Cotler et al., '16, '17

ETH reduces to RMT?

- For small $\omega \leq D/L^2$, $f(\omega)$ is constant and r_{nm} is GOE

  $\langle E_i|A|E_j\rangle = A^{\mathrm{eth}}\delta_{nm} + \Omega^{-1/2} f(\omega) r_{ij}$

  D'Alessio, Kafri, Polkovnikov, Rigol'15

- Gaussian distribution of r_{ii} and r_{ij}

  Beugeling, Moessner, Haque'14, ...

- ratio $\langle r_{ii}^2\rangle = 2\langle r_{ij}^2\rangle$

  AD and Liu, arxiv:1702.07722, Mondaini, Rigol'17

Expectation: ETH reduces to RMT at diffusion (Thouless) energy $|\omega| \leq E_{\mathrm{Th}} \equiv D/L^2$.

Thermalization – conventional picture

Diffusive system thermalizes within diffusion (Thouless) time $\tau \sim L^2$ necessary for the slowest diffusive modes to propagate across the system. After time $t \sim \tau$ the system is fully ergodic (and naively, ETH should reduce to RMT).

The key idea: dynamics of “slowest transport mode” constraints ΔE_{RMT}

New technical ingredient – deviation function

- connection between time evolution and linear algebra of $\hat{A}$

$$\int_{-\infty}^{\infty} dt \frac{\sin(t/T)}{\pi t} \langle \Psi | A(t) | \Psi \rangle = \langle \Psi | \begin{bmatrix} * & \diagdown & & 0 \\ \diagdown & * & & \\ & & \dots & \swarrow \\ & & * & \diagdown \\ 0 & \nearrow \diagdown & & * \end{bmatrix} | \Psi \rangle$$

$$2/T$$

- deviation function, arXiv:1702.07722

$$x(\Delta E) = \lambda_{\max} \left(\begin{bmatrix} 0 & & 0 \\ & \underbrace{[\dots]}_{2\Delta E}\}2\Delta E & \\ 0 & & 0 \end{bmatrix} \right)$$

Uniform bound on averaged time evolution

- Heuristic argument: after time t energies E_i, E_j, $|E_i - E_j|t \geq 1$ are mutually de-phased

$$\langle\Psi|A(t)|\Psi\rangle = \sum_{ij} C_i^* C_j A_{ij} e^{-i(E_i - E_j)t} \approx \sum_k \langle\Psi_k|A(t)|\Psi_k\rangle$$

from here, we have (naively):

$$|\langle\Psi|\delta A(t)|\Psi\rangle| \leq x(1/t)$$

- Conjecture: uniform bound on time-averaged dynamics, arxiv:1806.04187

$$\left|\int_{-\infty}^{\infty} dt \frac{\sin(t/T)}{t} \langle\Psi|\delta A(t)|\Psi\rangle\right| \leq 3\, x(1/t)$$

If observable is a Random Matrix

- form-factor $f^2(\omega) = \sum_j r_{ij}^2\, \delta(\omega - E_i + E_j)$

- assuming fluctuations r_{ij} are *independent* maximal eigenvalue of band random matrix is bounded by

$$x^2(\Delta E) \leq 8 \int_0^{2\Delta E} d\omega\, |f(\omega)|^2, \qquad A_{\Delta E} = \begin{bmatrix} * & \diagdown & & & 0 \\ \diagdown & * & & & \\ & & \cdots & & \swarrow \\ & & & * & \diagdown \\ 0 & & \underset{2\Delta E}{\nearrow} & \diagdown & * \end{bmatrix}$$

arXiv:1702.07722

- ◇ Gaussian Random Matrix, $f^2 = \text{const}$,
  $\Delta E(x) = x^2/(8f^2), \qquad x \propto \Delta E^{1/2}$

Upper bound on ΔE_{RMT} from transport

- for sufficiently large T, such that $T\Delta E_{\mathrm{RMT}} \geq 1$

$$\max_{\Psi}\left|\int dt\,\frac{\sin(\pi t/T)}{t\ \pi}\langle\Psi(t)|\delta A|\Psi(t)\rangle\right|^2 \leq \int dt\,\frac{\sin(\pi t/T)}{t\ \pi}\langle A(t)A(0)\rangle_{\beta}$$

this is a consistency condition for the observable restricted to an energy interval ΔE_{RM} to be a random matrix

Quasi-classical thermalization through transport

- a state Ψ with a macroscopic inhomogeneity of conserved quantity (energy); $\langle\Psi|\delta A(t)|\Psi\rangle$ remains of order one for Thouless time $t \sim \tau$, where $\delta A = A - A^{\rm eth}$

$$\langle\Psi|\delta A(t)|\Psi\rangle \sim e^{-t/\tau}$$

  time τ grows polynomially with the system size L, $\tau \sim L^2/D$

- the deviation $\delta A(t)$ averaged over time T

$$\int dt \langle\Psi|\delta A(t)|\Psi\rangle \frac{\sin(\pi t/T)}{\pi t} \approx \frac{1}{T}\int_0^T dt \langle\Psi|\delta A(t)|\Psi\rangle \sim \frac{\tau}{T}$$

Behavior of autocorrelation function

When the system is chaotic (non-integrable, not MBL), two-point function is diffusive

$$\langle A(t)A(0)\rangle_{\beta(E)} = \int d\omega\, f^2(E,\omega)\, e^{-i\omega t}$$

- for late times, but before diffusion ends $t < \tau = L^2/D$, diffusive Green's function $\sim 1/t^{d/2}$
- after diffusion (Thouless) time $\tau = L^2/D$ Green's function saturates, $f^2(\omega)$ becomes a plateau for small $|\omega| \leq 1/\tau$

$$\langle E|A(t)A(0)|E\rangle \sim \begin{cases} 1/\sqrt{t} & t \leq \tau = L^2/D \\ 0 & t \geq \tau = L^2/D \end{cases}$$

autocorrelation function is well-behaved in $L \to \infty$ limit

Upper bound on $\Delta E_{\rm RMT}$ from slow states

- for sufficiently large T, such that $T\Delta E_{\rm RMT} \geq 1$

$$\max_{\Psi}\left|\int dt\,\frac{\sin(\pi t/T)}{t\;\pi}\langle\Psi(t)|\delta A|\Psi(t)\rangle\right|^2 \leq \int dt\,\frac{\sin(\pi t/T)}{t\;\pi}\langle A(t)A(0)\rangle_\beta$$

- LHS behaves as $(\tau/T)^2$, it grows polynomially with L
- RHS (naively) behaves as $1/\sqrt{T}$, which is L-independent
  a bit more carefully, RHS is $\sqrt{\tau}/T$ for $T > \tau$

$$\left(\frac{\tau}{T}\right)^2 \leq \frac{\sqrt{\tau}}{T} \Rightarrow T \geq \tau^{3/2} \sim L^3$$

Result and interpretation

$$\Delta E_{RM}^{-1} = T \geq L^3 \gg \tau = L^2/D$$

- scale of applicability of Random Matrix Theory to describe a local observable is parametrically smaller than Thouless (diffusion) energy D/L^2
- potential interpretation: T is the time scale when $\langle\Psi|\delta A(t)|\Psi\rangle \sim e^{-t/\tau}$ becomes as small as exponentially small late time fluctuations, $|\langle\Psi|\delta A(t)|\Psi\rangle| \sim e^{-S/2}$ (ETH)

$$\frac{T}{\tau} \sim S \sim L \Rightarrow T \sim \tau L \sim L^3$$

Conclusions

- Macroscopic transport constraints off-diagonal elements of ETH ansatz. What is the right language to describe statistical properties of matrix elements before Random Matrix Theory is applicable?

- The Random Matrix scale $T = \Delta E_{\mathrm{RMT}}^{-1}$, when ETH reduces to Random Matrix Theory is parametrically longer than the diffusion (Thouless) time

- Conjecture: Random Matrix Theory describes expectation values $\langle\Psi|A(t)|\Psi\rangle$ at late times $t > T \sim L^3$. Then T is the time of "universality"