Corrections to scaled limits in random matrix theory

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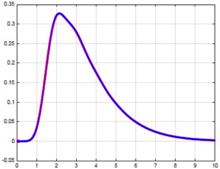
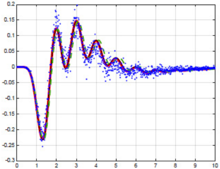
- Riemann zero data and Painléve transcendents
- Finite size corrections to Riemann zero data
- Finite size corrections in RMT scaling limits





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Gap probabilities and Painléve transcedents

Painléve transcendents — solutions of one of six nonlinear 2nd order differential equations, each with the property that no moveable singular points are essential singularities.

There is a (non-autonomous) Hamiltonian theory (Malmquist). The Hamiltonian satisfies a so-called Painléve equation in sigma form, e.g. σPII, $(\sigma'')^2 + 4\sigma'\Big((\sigma')^2 - t\sigma' + \sigma\Big) - a^2 = 0$.

Since the work of the Kyoto school (1980) it has been known that **gap probabilities** in random matrix theory permit evaluations in terms of sigma Painléve transcendents, e.g. (F. & Witte, 2004) $p(s;0;\xi)|_{\beta=2} = \frac{\pi^2}{3} s^2 \exp \int_0^{2\pi s} u(t;\xi)\,\frac{dt}{t}$, where

$$\Big(su''(s)\Big)^2 + \Big(su'(s) - u(s)\Big)\Big(su'(s) - u(s) - 4 + 4(u'(s))^2\Big) - 16\Big(u'(s)\Big)^2 = 0$$

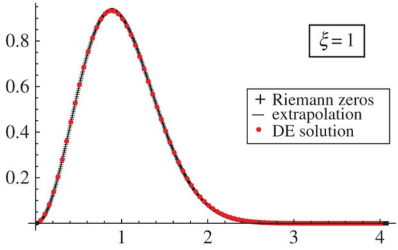
subject to the boundary condition

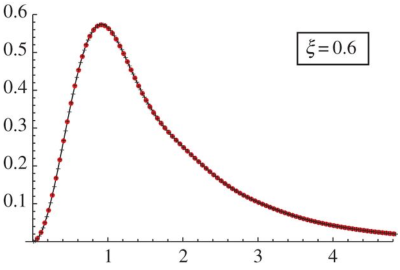
$$u(s;\xi) \underset{s\to 0^+}{\sim} -\frac{1}{15}s^2 + \mathrm{O}(s^4) - \frac{\xi}{8640\pi}\Big(s^5 + \mathrm{O}(s^7)\Big).$$

Riemann zero data

Montgomery-Odlyzko law: the statistics of the large **Riemann zeros** have the same distribution as the bulk eigenvalues of a large **complex Hermitian** random matrix.

Veracity can be probed using a high precision, big data set due to Odlyzko ($\sim$ 2000). The data set begins with zero number $10^{23} + 985,531,550$, and lists the next 10^9.
This occurs at the point $s = 1/2 + iE$ in the complex s-plane with E equal to
$1306643440879362112002 7.3961465854 \approx 1.30664344 \times 10^{22}$.





A grand challenge/ suggestion — underlying determinantal point process for RZ?

Keating and Snaith (2000) hypothesised a $U(N)$ random matrix model for the leading corrections. Further developed by Bogomolny and collaborators.

The eigenvalues $e^{i\theta} = e^{i2\pi x/N}$ of $U(N)$ matrices form a determinantal point process:

$$\rho_{(k)}(x_1, \ldots, x_k) = \det\Big[K_N(x_i, x_j)\Big]_{i,j=1}^{k}, \quad K_N(x, y) = \frac{1}{N}\frac{\sin\pi(x-y)}{\sin\pi(x-y)/N}.$$

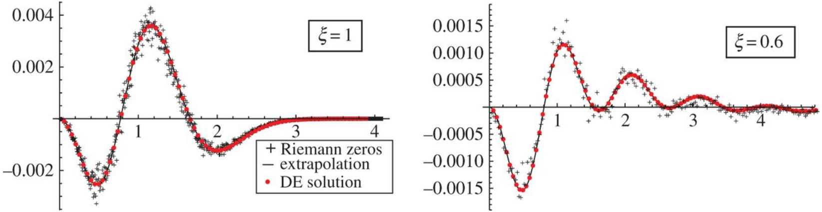
With $N = \frac{1}{\sqrt{12\Lambda}}\log\left(\frac{E}{2\pi}\right)$, $\alpha = 1 + \frac{C}{\log(E/2\pi)}$, all results known to date are consistent with

$$\rho^{\mathrm{RZ}}_{(k)}(x_1, \ldots, x_k) = \det\Big[K^{\mathrm{RZ}}_N(x_i, x_j)\Big]_{i,j=1}^{k}$$

where

$$K^{\mathrm{RZ}}_N(x, y) = \frac{\sin\pi(x-y)}{\pi(x-y)} + \frac{\pi(x-y)\sin(\pi(2\alpha-1)(x-y))}{6N^2} + \mathrm{O}\Big(\frac{1}{N^4}\Big).$$

The implied RMT challenge: a theory of finite size corrections



Results from F. & Witte (2004) give

$$\frac{2\pi}{N} p^N(2\pi s/N; 0; \xi) = \frac{1}{3}(N^2 - 1)\sin^2\frac{\pi s}{N}\exp\left(-\int_0^{\pi s/N} V(\cot\phi; \xi)\, d\phi\right)$$

where V satisfies a particular $\tilde{\sigma}$PVI equation. Hence

$$\frac{2\pi}{N} p^N(2\pi s/N; 0; \xi) = \frac{\pi^2 s^2}{3}\exp\left(\int_0^{2\pi s}\frac{u^{(0)}(X)}{X}\, dX\right) \times \left(1 - \frac{1}{N^2} - \frac{\pi^2 s^2}{3N^2} + \frac{1}{N^2}\int_0^{2\pi s}\frac{u^{(1)}(X)}{X}\, dX + \mathrm{O}\left(\frac{1}{N^4}\right)\right).$$

A second order linear DE with σPV coefficients

We find that $u^{(1)}(X)$ satisfies the second order, linear differential equation

$$\tilde{A}(s)y''(s) + \tilde{B}(s)y'(s) + \tilde{C}(s)y(s) = \tilde{D}(s),$$

where, with $u(s) = u^{(0)}(s)$, $\tilde{A}(s) = 8s^2 u''(s)$. The other coefficients are also explicit polynomials in $\{u(s), u'(s), u''(s), s\}$. The equation must be solved subject to the $s \to 0^+$ boundary condition

$$u^{(1)}(s) = \frac{4}{15}s^2 - \frac{13}{6300}s^4 + \frac{\xi}{1728\pi}s^5 + \mathrm{O}(s^6).$$

Universality results give that *all* random matrices in a certain class (for example complex Hermitian with independent entries from the same zero mean, unit variance distribution) have, after scaling, the same large N statistical properties. What about finite N corrections?

Setting up the question in the bulk

Option 1

- Choose a unitary invariant ensemble, or a complex Wigner ensemble. The simplest choice would be to consider GUE matrices.
- For complex Wigner ensembles the eigenvalue density will be to leading order given by the Wigner semi-circle law, supported to leading order on $(-\sqrt{2N}, \sqrt{2N})$. Use this density to **unfold** the eigenvalues.
- Now compute the averaged spacing distribution for some finite fraction of the eigenvalues about the origin.
- The task is to compute the large N form of this averaged spacing distribution.

Option 2

- Instead of averaging over a finite fraction of the eigenvalues, ask specifically about the spacing between, say, the two middle eigenvalues.

Gap probabilities and correlations

Let $E(n; J)$ denote the probability that the interval J contains exactly n eigenvalues. Define the generating function $G(J; \xi)$ by

$$G(J; \xi) := \sum_{n=0}^{\infty} (1-\xi)^n E(n; J).$$

Specifically, $G(J; \xi)|_{\xi=1} = E(0; J)$.

In terms of the k-point correlations $\rho_{(k)}$,

$$G(J; \xi) = 1 + \sum_{k=1}^{\infty} \frac{(-\xi)^k}{k!} \int_{a_1}^{a_2} dx_1 \cdots \int_{a_1}^{a_2} dx_k \, \rho_{(k)}(x_1, \ldots, x_k)$$

Remarks

- The sums terminate at N for a finite system.
- Can interpret $G(J; \xi)$ as the probability that the interval J is free of eigenvalues, in the setting that each eigenvalue has been deleted independently with probability $(1-\xi)$.

Spacing distributions and correlations

The PDF for the event that, given there is an eigenvalue at a_1, the next eigenvalue to the right is at a_2, is given by

$$\begin{aligned} p((a_1, a_2)) &= -\frac{1}{\rho_{(1)}(a_1)} \frac{\partial^2}{\partial a_1 \partial a_2} E(0; (a_1, a_2)) \\ &= \frac{1}{\rho_{(1)}(a_1)} \rho_{(2)}(a_1, a_2) + \cdots \end{aligned}$$

Note that this is relevant to Option 1 via the averaged quantity

$$\bar{p}(s) := \int p((a, a+s))\, da = \int \frac{1}{\rho_{(1)}(a)} \rho_{(2)}(a, a+s)\, da + \cdots$$

Difficult. Ask instead a related question: what are the properties of the finite size correction at the soft edge (i.e. neighbourhood of the largest eigenvalue).

Soft edge leading corrections

We have

$$p_{\max}(s) = \frac{d}{ds}\Big(\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_s^\infty dx_1 \cdots \int_s^\infty dx_k \, \rho_{(k)}(x_1, \dots, x_k) \Big)$$
$$= \rho_{(1)}(s) - \int_s^\infty \rho_{(2)}(s, x) \, dx + \cdots$$

Introduce the scaled variable

$$s_{N,t} = \begin{cases} \sqrt{2N} + t/\sqrt{2}N^{1/6}, & \text{GUE} \\ 4N + 2a + 2(2N)^{1/3} t, & \text{LUE} \end{cases}$$

F. & Trinh (2017) have shown that

$$p_{\max,N}(t;\xi) = p_{\max,\infty}(t;\xi) + \frac{1}{N^{2/3}} \tilde{p}(t;\xi) + \mathrm{O}\Big(\frac{1}{N}\Big).$$

We know that, with $q'' = sq + 2q^3$,

$$p_{\max,\infty}(t;\xi) = \frac{d}{dt} \det(\mathbb{I} - \xi \mathbb{K}^{\mathrm{Airy}}_{(t,\infty)})$$
$$= \frac{d}{dt} \exp\Big(- \int_t^\infty (x - t) q^2(x;\xi) \, dx \Big)$$

Weak universality?

The function $\tilde{p}(t;\xi)$ can be characterised as the solution of a 2nd order linear differential equation with σPII coefficients. It is *different* for the GUE and LUE.

Note subtlety in relation to the Laguerre case $s_{N,t} = 4N + 2(2N)^{1/3}t + 2a$. What would happen if instead we set $s_{N,t} = 4N + 2(2N)^{1/3}t$?

$$p_{\max,N}(t+\frac{c}{N^{1/3}};\xi) = p_{\max,\infty}(t;\xi)+\frac{c}{N^{1/3}}\frac{d}{dt}p_{\max,\infty}(t;\xi)+\mathrm{O}\Big(\frac{1}{N^{2/3}}\Big).$$

Hence, a **non-optimal** choice of scaling can be detected by the "leading" correction term being related to the scaled limit as a derivative.

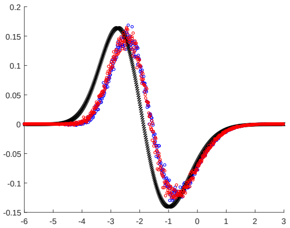
Question: Can a soft edge scaling variable always be chosen so that the optimal correction term is $\mathrm{O}\Big(\frac{1}{N^{2/3}}\Big)$?

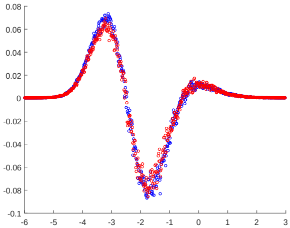
A particular Wigner ensemble

Define X so that its entries are chosen independently and uniformly from the set of four values $\frac{1}{\sqrt{2}}(\pm 1 \pm i)$.

In terms of X define the Hermitian matrix $Y = \frac{1}{2}(X + X^{\dagger})$. Off diagonal entries have mean zero and variance $\frac{1}{2}$. Hence to leading order largest eigenvalue is equal to $\sqrt{2N}$.

Plot histograms associated with the scaled random variable $t = \sqrt{2}N^{1/6}(\lambda_{\max} - \sqrt{2N})$, minus $p_{\max,\infty}(t)$ multiplied by $N^{1/3}$ (left). Except for shifting by $c = 1/2$, this appears to be the graph for $\frac{d}{dt}p_{\max,\infty}(t)$. Replacing t by $t - 1/(2N^{1/3})$ leads to a $N^{-2/3}$ correction.





Future work

Study optimal soft edge scaling for the density in Gaussian and Laguerre beta ensemble with beta even. This is possible due to certain duality formulas, e.g.

$$\left\langle \prod_{l=1}^{N} \left(x - \sqrt{\frac{2}{\beta}} x_l \right)^n \right\rangle_{\mathrm{GE}_{\beta,N}} = \left\langle \prod_{j=1}^{n} (x - i x_j)^N \right\rangle_{\mathrm{GE}_{4/\beta,n}}.$$

Find in the Gaussian case that choosing

$$\lambda = \sqrt{2N} + \frac{1}{\sqrt{2}N^{1/6}} \left(x + \left(\frac{1}{2} - \frac{1}{\beta} \right) \frac{1}{N^{1/3}} \right)$$

gives optimal $O(N^{-2/3})$ correction. For the Laguerre case, the same (a dependent) scaling as for $\beta = 2$ is optimal.

For the unitary invariant ensemble with potential $e^{-Nx^{2m}}$ a result of Dieft and Gioev establishes m dependent values of b and γ so that $\lambda = b + \frac{x}{N^{2/3}\gamma}$ gives optimal $O(N^{-2/3})$ correction.

Presently studying similar questions at the hard edge.