What drives transient behaviour in complex systems?

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Random Matrices, Integrability and Complex Systems

Yad Hashmona

based on Phys. Rev. E 96, 022316
Outline

- Motivation(s)
- May-Wigner and beyond
- Generators of transient behaviour
Choose a dynamical complex system
Description: \[
\frac{dx_i}{dt} = f_i(x_1...x_N), \quad i = 1...N
\]
Examples
1. Lotka-Volterra \[ f_i = x_i(1 - \sum_j \alpha_{ij}x_j) \]
2. deep learning \[ f_i = \tanh(\sum_j C_{ij}x_j) \]
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- Simplest case – stability analysis $x_i = x_i^* + y_i$ for $f_i(x^*) = 0$
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  \frac{dy_i}{dt} = \sum_{j=1}^{N} M_{ij}y_j, \quad M_{ij} = \partial_{x_j} f_i(x) |_{x=x^*}
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![Graph showing intermediate and late time behaviors](image-url)
Motivation

- Eigenvalues of $M$ – late time analysis
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- What about intermediate or early times? Hard in general...

\[ |y(t)|^2 = \sum_i |y_i|^2 \]

Two indicators:
- Amplification: $A = \max_{t \geq 0} |y(t)|^2 |y_0|^2$
- Reactivity: $R = \frac{1}{|y_0|^2} \lim_{t \to 0} \int |y(t)|^2 \, dt$,

where $y_0$ is the initial condition.
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- A tangible feature – **transient (growth) behaviour**

\[ |y(t)|^2 = \sum_i y_i^2 \text{ is the norm.} \]
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Motivation

- For a linear model the solution (bra-ket notation):

\[ |y(t)⟩ = e^{Mt} |y_0⟩, \]

The norm is

\[ ⟨y(t)|y(t)⟩ = ⟨y_0|e^{MT}e^{Mt}|y_0⟩, \]

and so the reactivity

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- When does it show up? $M$ must be **asymmetric**. Not concrete enough!

Does it matter? It does, at least for fluids, brains and ecosystems (Neubert & Caswell 1997 – rainforest compartment model)

Consider the following mass–energy flow diagram:
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The model is of the same form:

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\frac{dy_i}{dt} = \sum_{j=1}^{9} M_{ij}y_j,
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- the matrix \( M \):

<table>
<thead>
<tr>
<th>Compartment and number</th>
<th>1</th>
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<th>3</th>
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<th>6</th>
<th>7</th>
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- all eigenvalues of \( M \) are negative – stable!
Motivation

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Plot of $\max_{y_0} \frac{|y(t)|^2}{|y_0|^2}$ in log-plot:

a giant amplification
May-Wigner model

(Lord May 1973) Consider a model:

\[
\frac{dy_i}{dt} = \sum_{j=1}^{N} M_{ij} y_j,
\]

\(M\) will be of size \(N \times N\):

\[M = -\mu 1_N + X\]

where \(\mu > 0\) (what is it?) and \(X\) is random drawn from:

\[P(X)[dX] \sim \exp \left( -\frac{N}{2\sigma^2} \text{Tr}X^T X \right) [dX],\]
May-Wigner model

- Late time behaviour – eigenvalues of $M$. Asymptotic ($N \to \infty$) density:

$$\rho_M(x, y) = \frac{1}{\pi \sigma^2} \theta \left( \sigma^2 - (x + \mu)^2 - y^2 \right),$$

or the circular law (only translated)

$$\begin{array}{cccccc}
\mu & \text{density} & \rho_M & \text{unstable} & \text{stable} \\
-3 & -2 & -1 & 0 & 1 \\
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- System is stable if $\mu < \mu_s$ and unstable when $\mu > \mu_s$ for $\mu_s = \sigma$.

Phase space is one-dimensional:
Beyond May-Wigner model

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- Treating initial conditions:
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  2. Maximized \( O_{max} = \max_{y_0} O \),

\[ R_{max} = \max_{y_0} \langle y_0 | (M^T + M) | y_0 \rangle \langle y_0 | y_0 \rangle = \lambda_{max}(M^T + M) \]

In the asymptotic \( N \to \infty \) limit:
\[ \lim_{N \to \infty} \langle R_{max} \rangle_{X} = -2 \mu + 2 \mu_T, \quad \mu_T = \sqrt{2} \sigma \]
which is found by using the Wigner's semicircle law for the eigenvalues of \( M^T + M \).
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\begin{align*}
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Voilà, new regime!

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- Averaged variant is found by completing the square
  \[ \overline{g_{av}}(r) = \frac{1}{\sqrt{2\pi\sigma_R^2}} e^{-\frac{(r+2\mu)^2}{2\sigma_R^2}}, \quad \sigma_R^2 = 4\sigma^2/N \]
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- The extreme variant is the GOE Tracy-Widom distribution

  \[ \overline{g_{max}}(r) = \langle \delta \left( r + 2\mu - \lambda_{max}(X^T + X) \right) \rangle_X = \frac{d}{dr} F_{N,\beta=1} \left( \frac{\sqrt{N}}{\sigma} \left( \mu + \frac{r}{2} \right) \right) \]
The abundance of transient trajectories:

\[ N_{\text{max}} = \int_{0}^{\infty} g_{\text{max}}(r) dr, \quad N_{\text{av}} = \int_{0}^{\infty} g_{\text{av}}(r) dr. \]

\[ N_{\text{max}}(\mu) = 1 - F_{N,\beta=1} \left( \frac{\sqrt{N\mu}}{\sigma} \right), \quad N_{\text{av}}(\mu) = \frac{1}{2} \text{erfc} \left( \frac{\sqrt{N\mu}}{\mu_T} \right). \]
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![Graph showing the relationship between \(N_{\text{max}}\) and \(N_{\text{av}}\) with \(\mu\) as the independent variable, indicating stable and transient regimes.]

- transient trajectories are (potentially) present in the whole transient regime \(\mu \in (\mu_S, \mu_T)\) as shown by the behaviour of \(\overline{N}_{\text{max}}\), they are otherwise uncommon as dictated by \(\overline{N}_{\text{av}}\).
Transient behaviour generators


\[ X = O (Z + T) O^T, \]

where \( T \) is block-upper triangular (eigenvectors), \( Z \) block-diagonal (eigenvalues) and \( O \) orthogonal.

Go back to reactivity \( R \) and compute the \( X \) average as decomposed

\[ \langle R \rangle_X = -2\mu + 1 \langle y_0 | y_0 \rangle (\langle y_0 | Z^T + Z \rangle_X | y_0 \rangle + \langle y_0 | T^T + T \rangle_X | y_0 \rangle) \]

In May-Wigner model, both terms vanish.

Introduce a modification of the model – fix \( T \) and leave the \( Z \) unchanged so that only one vanishes.
Transient behaviour generators

- To understand, consider Schur decomposition:

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$$\tilde{P}(X; T_0)[dX] \sim \delta(T - T_0)P(X)[dX],$$
Transient behaviour generators

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- Seems artificial but has two properties:
  1. Does not spoil stability (eigenvalues stay in place)
  2. Does modify reactivity (transient regime!)
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...or a (not so distant) echo of eigenvectors.
Transient behaviour generators

- Phase space is now two-dimensional with proper transient regime:

\[ \frac{\mu}{\sigma}, \frac{\tau}{\sigma} \]

- Transient
- Non-transient

\( |y(t)|^2 \) vs. \( t \)

(a) unstable
(b) transient
(c) non-transient
(d) stable

\( \tau = 2\mu \)

Normal X
Recap and Future

Recap:

- Transient behaviour is an early time phenomenon abundant in real-life systems.
- May-Wigner model contains a regime of parameters where transient dynamics is present although rare.
- Transient trajectories are generated by eigenvector degrees of freedom.

Future:

- Reactivity is not an exact measure of transient behaviour.
- What about $t_{\text{max}}$ or amplification?
- Statistics of the norm $|y(t)|^2$ (variance and beyond).
- Echoes of transient behaviour in the chaotic phase.
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- Transient behaviour is an early time phenomenon abundant in real-life systems
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