What drives transient behaviour in complex systems?

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Random Matrices, Integrability and Complex Systems

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Outline

- Motivation(s)
- May-Wigner and beyond
- Generators of transient behaviour

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- Choose a dynamical complex system
  Description: $\frac{dx_i}{dt} = f_i(x_1 ... x_N), \qquad i = 1 ... N$
  Examples
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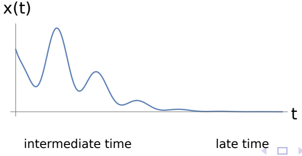

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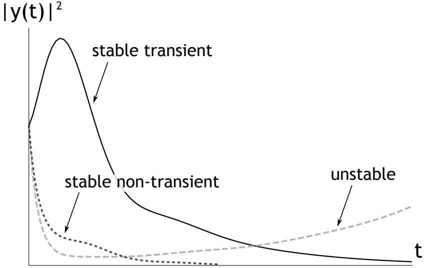

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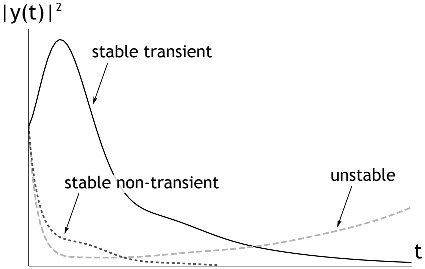

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- Two indicators **amplification** and **reactivity**:

$$A = \max_{t \geq 0} \frac{|y(t)|^2}{|y_0|^2}, \qquad R = \frac{1}{|y_0|^2} \lim_{t \to 0} \frac{d|y(t)|^2}{dt},$$

where y_0 is the initial condition.

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- we have $A = \max_{t\geq 0}\left(1 + Rt + \mathcal{O}(t^2)\right)$.

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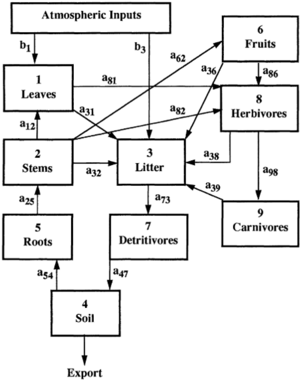
- Consider the following mass–energy flow diagram:



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- the matrix M:

TABLE 2. Transfer matrix for elemental dynamics in a Panamanian tropical forest (cf. McGinnis et al. 1969).

Compartment and number		1	2	3	4	5	6	7	8	9
Leaves	1	−1.5622†	0.6685							
Stems	2		−0.7119			2.5632				
Litter	3	1.4627	0.0364	−6.4091			1.1446		55.8201	17.2972
Soil	4				−0.0222			315.9443		
Roots	5				0.0201	−2.5632				
Fruits and flowers	6		0.0070				−2.0348			
Detritivores	7			6.4091				−315.9443		
Herbivores	8	0.0995					0.8902		−62.6458	
Carnivores	9								6.8257	−17.2972

† All entries are in units of yr^{-1}.

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- all eigenvalues of M are negative – stable!

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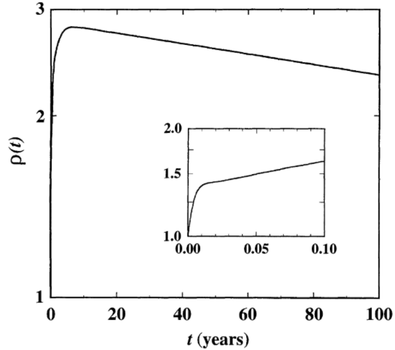
Plot of $\max_{y_0} \frac{|y(t)|^2}{|y_0|^2}$ in log-plot:



a giant amplification

May-Wigner model

- (Lord May 1973) Consider a model:

$$\frac{dy_i}{dt} = \sum_{j=1}^{N} M_{ij} y_j,$$

- M will be of size $N \times N$:

$$M = -\mu 1_N + X$$

where $\mu > 0$ (what is it?) and X is random drawn from:

$$P(X)[dX] \sim \exp\left(-\frac{N}{2\sigma^2}\mathrm{Tr}X^T X\right)[dX],$$

May-Wigner model

- Late time behaviour – eigenvalues of M. Asymptotic ($N \to \infty$) density:

$$\rho_M(x, y) = \frac{1}{\pi\sigma^2}\theta\left(\sigma^2 - (x + \mu)^2 - y^2\right),$$

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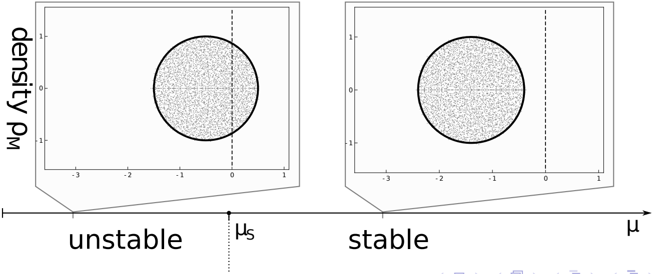
- System is stable if $\mu < \mu_s$ and unstable when $\mu > \mu_s$ for $\mu_s = \sigma$. Phase space is one-dimensional:



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$$R_{\max} = \max_{y_0} \frac{\langle y_0 | (M^T + M) | y_0 \rangle}{\langle y_0 | y_0 \rangle} = \lambda_{\max} \left(M^T + M \right) .$$

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- In the asymptotic $N \to \infty$ limit:

$$\lim_{N \to \infty} \langle R_{\max} \rangle_X = -2\mu + 2\mu_T, \qquad \mu_T = \sqrt{2}\sigma$$

which is found by using the Wigner's semicirlce law for the eigenvalues of $M^T + M$.

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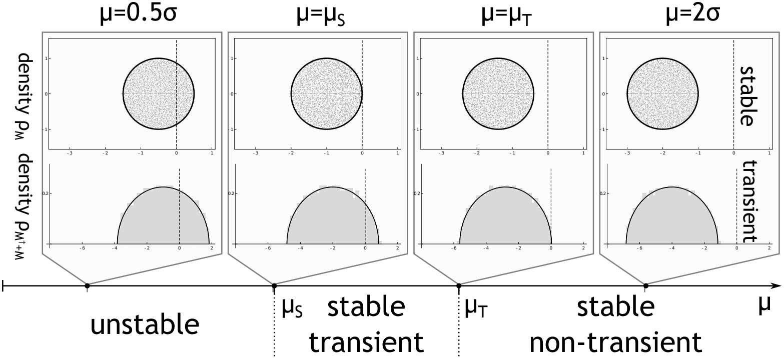

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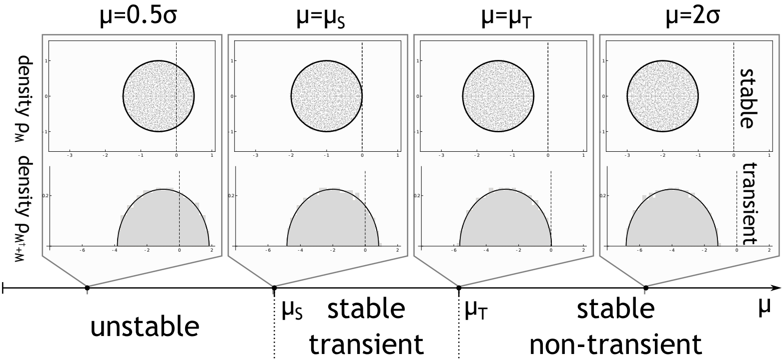

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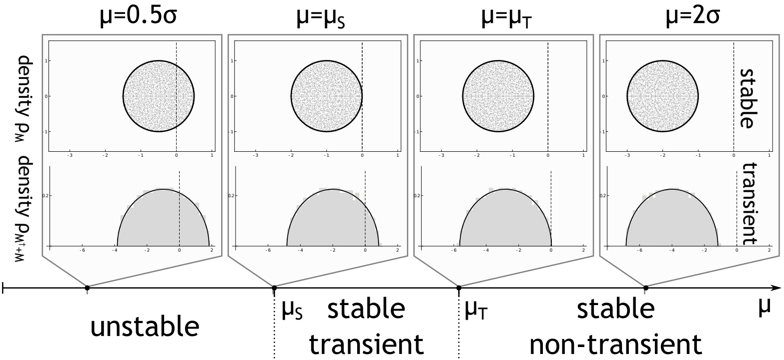

- Voilà, new regime!
- But... We see that $\overline{R_{\mathrm{av}}} = -2\mu < 0$. What is then its nature?

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- The extreme variant is the GOE Tracy-Widom distribution

$$\overline{g_{\max}}(r) = \left\langle \delta\left(r + 2\mu - \lambda_{\max}(X^T + X)\right)\right\rangle_X = \frac{d}{dr} F_{N,\beta=1}\left(\frac{\sqrt{N}}{\sigma}\left(\mu + \frac{r}{2}\right)\right)$$

- The abundance of transient trajectories:

$$\overline{N_{\text{max}}} = \int_0^\infty \overline{g_{\text{max}}}(r)dr, \quad \overline{N_{\text{av}}} = \int_0^\infty \overline{g_{\text{av}}}(r)dr.$$

$$\overline{N_{\text{max}}}(\mu) = 1 - F_{N,\beta=1}\left(\frac{\sqrt{N}\mu}{\sigma}\right), \overline{N_{\text{av}}}(\mu) = \frac{1}{2}\text{erfc}\left(\sqrt{N}\frac{\mu}{\mu_T}\right).$$

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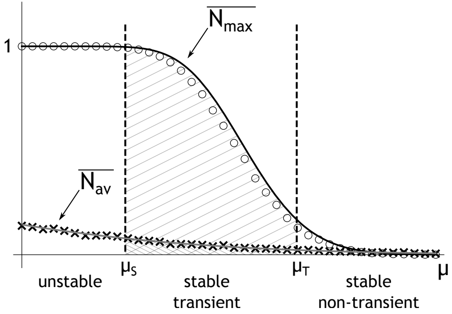

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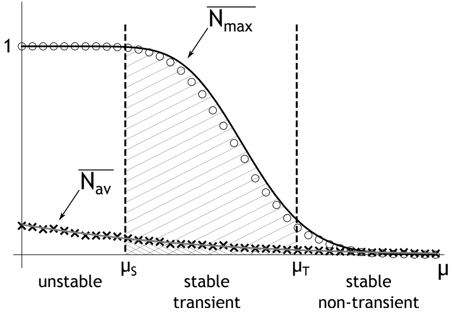

- **transient trajectories** are **(potentially) present** in the whole transient regime $\mu \in (\mu_S, \mu_T)$ as shown by the behaviour of $\overline{N_{\max}}$, they are otherwise **uncommon** as dictated by $\overline{N_{\mathrm{av}}}$.

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- Go back to reactivity R and compute the X average as decomposed into two parts

$$\langle R \rangle_X = -2\mu + \frac{1}{\langle y_0 | y_0 \rangle}\Big(\Big\langle y_0 \Big| \Big\langle Z^T + Z \Big\rangle_X \Big| y_0 \Big\rangle + \\ + \Big\langle y_0 \Big| \Big\langle T^T + T \Big\rangle_X \Big| y_0 \Big\rangle \Big)$$

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- In May-Wigner model, both terms vanish.

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- Go back to reactivity R and compute the X average as decomposed into two parts

$$\langle R \rangle_X = -2\mu + \frac{1}{\langle y_0 | y_0 \rangle} \Big(\Big\langle y_0 \Big| \Big\langle Z^T + Z \Big\rangle_X \Big| y_0 \Big\rangle + \\ + \Big\langle y_0 \Big| \Big\langle T^T + T \Big\rangle_X \Big| y_0 \Big\rangle \Big)$$

- In May-Wigner model, both terms vanish.
- Introduce a modification of the model – fix T and leave the Z unchanged so that only one vanishes.

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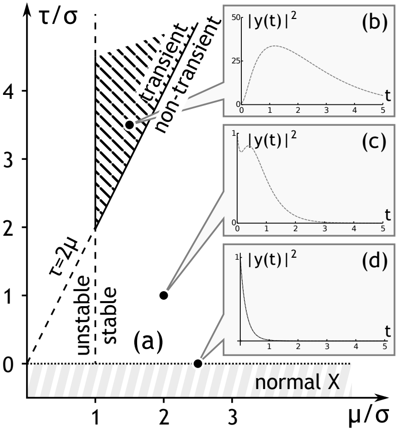
- ...or a (not so distant) echo of eigenvectors

Transient behaviour generators

- Phase space is now two-dimensional with proper transient regime:



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- Transient trajectories are generated by eigenvector degrees of freedom

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Future:

- Reactivity is not an exact measure of transient behaviour
- What about $t_{\max}$? or amplification?
- Statistics of the norm $|y(t)|^2$ (variance and beyond)
- Echoes of transient behaviour in the chaotic phase