Quantum chaos versus quantum complexity

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Outline of the talk

– Motivation: Chaos versus complexity.

– Integrable spin chains. Classical chaos.

– Integrable spin chains. Quantum chaos.

Duisburg collaborators:
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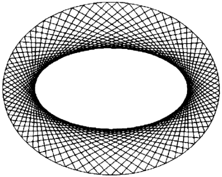
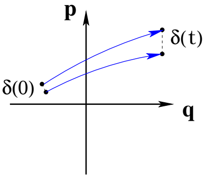
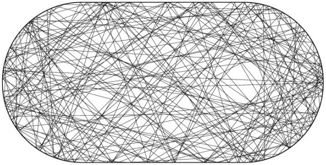
Classical Chaos







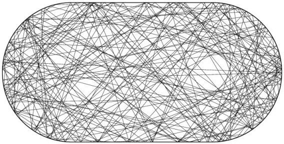
Integrable Hamiltonian systems. For 2N dimensional system, there are N integrals of motion $\Longrightarrow$

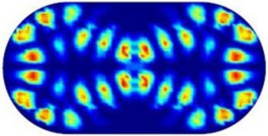
Dynamics are "regular". No ergodicity, $\delta(t) \sim t$. **Number of periodic orbits grows (at best) algebraically with** t

Hyperbolic Hamiltonian systems. Exponential sensitivity to initial conditions: $\delta(t) \sim \delta(0)e^{\lambda t}$ $\Longrightarrow$

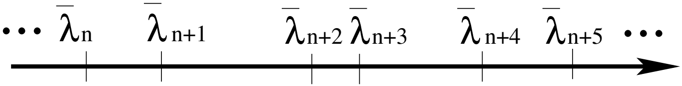
Dynamics are "chaotic". Only energy is conserved, Ergodicity. **Number of periodic orbits grows exponentially**

Quantum spectrum





Quantum: $-\Delta\varphi_n = \lambda_n\varphi_n,\ \varphi_n|_{\partial\Omega} = 0 \qquad \varphi_n \in L^2(\Omega)$



Hamiltonian Systems $\Longleftrightarrow$ **Unitary evolution** $U(t) = e^{-\frac{i}{\hbar}Ht}$

Statistics of $\bar{\lambda}_n = \lambda_n/\Delta$ (Δ-mean level spacing):

$$\mathrm{Tr}\, U(t) = \sum_n \exp\left(-it\lambda_n/\hbar\right), \qquad K(t) \sim \langle|\mathrm{Tr}\, U(t)|^2\rangle?$$

Spectral statistics

RMT approach. Substitute $U(t)$ by ensemble of unitary random matrices (from the same symmetry class):

$$U(t) \to U \in \text{ CUE, COE, CSE}$$

$+$ Hope for the best

E. Wigner 1955; G.Casati, et al. 1980; O. Bohigas, et al. 1984

Semiclassical approach $\hbar \to 0$.

Gutzwiller/Berry-Tabor **trace formula**:

$$\mathrm{Tr}\, U(t) \sim \sum_{\gamma} \mathcal{A}_{\gamma} \exp\left(\frac{i}{\hbar} S_{\gamma}(t)\right)$$

$+$ Periodic orbits correlations

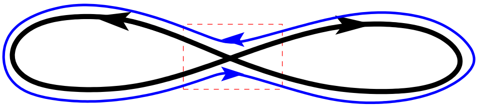
M.V. Berry, M. Tabor 1977; M. Berry 1985

Fully chaotic systems

Bohigas-Giannoni-Schmit conjecture:
On the scales of mean level spacing $K(t)$ **is universal function**, provided by RMT ensemble from the same symmetry class

$$K(t) = \frac{1}{N}\sum_{n=1}^{N}\sum_{m=1}^{N} e^{-\frac{it}{\hbar}(\lambda_n-\lambda_m)} \sim \sum_{\gamma,\bar{\gamma}} \mathcal{A}_\gamma \mathcal{A}_{\bar{\gamma}} e^{\frac{i}{\hbar}(S_\gamma - S_{\bar{\gamma}})}$$

$\Longrightarrow$ **Actions of periodic orbits must correlate!**
M. Berry 1985, N. Argaman, et. al 1993



Sieber–Richter pairs

Taking into account all possible structures $\Longrightarrow$ **Full RMT result**
M. Sieber K. Richter 2001, S. Muller, et.al 2004

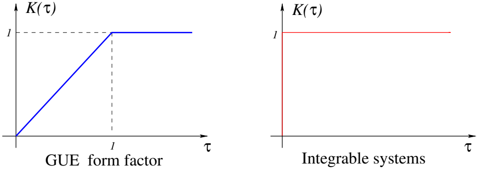
Integrable systems

$$K(t) \sim \sum_{\gamma,\bar{\gamma}} \mathcal{A}_\gamma \mathcal{A}_{\bar{\gamma}} e^{\frac{i}{\hbar}(S_\gamma - S_{\bar{\gamma}})}$$

Same expression, but **periodic orbits** are “rare“, **do not correlate**

$$\Updownarrow$$

Eigenvalues λ_n, $n = 1,,2\ldots$ **do not correlate.** Poissonian statistics M.V. Berry, M. Tabor 1977



End of the story?

Many-body integrable systems

Why it matters?

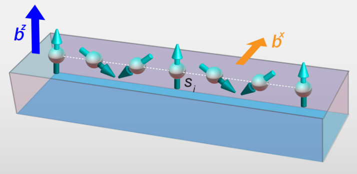
Two parameters - time t and number of particles N. Number of periodic orbits (PO) γ, their actions S_γ grow with both t and N.
B. G., V. Al. Osipov, Nonlinearity 29 (2016)

$$K(t) \sim \sum_{\gamma,\bar{\gamma}} \mathcal{A}_\gamma \mathcal{A}_{\bar{\gamma}} e^{\frac{i}{\hbar}(S_\gamma - S_{\bar{\gamma}})}$$

A) Can we have non-trivial PO correlations in integrable systems?

B) How do they affect spectral correlations, $K(t)$?

Toy model – Integrable ZZ spin chain



$$\hat{H} = \frac{J}{(s+1/2)^2} \sum_{n=1}^{N} \underbrace{\hat{S}_n^z \hat{S}_{n+1}^z}_{\text{interaction}} + \underbrace{\mu(\hat{S}_n^z)^2 + f(\hat{S}_n^z)}_{\text{ext. field}},$$

$$\hat{\boldsymbol{S}}_n = (\hat{S}_n^x, \hat{S}_n^y, \hat{S}_n^z), \qquad \hat{\boldsymbol{S}}_n^2 = s(s+1)$$

The dimension of the Hilbert space is **enormous** $(2s+1)^N$.
But the system is trivially integrable. What can we say about

$$K(t) = \frac{1}{(2s+1)^N} \left\langle |\mathrm{Tr}\, U(t)|^2 \right\rangle, \quad U(t) = e^{i(s+1/2)t\hat{H}} \ ?$$

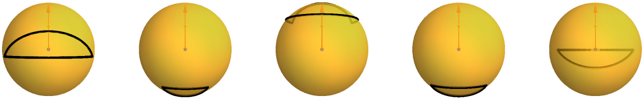
Semiclassical limit $s + 1/2 = 1/\hbar_{eff} \to \infty$

Classical system – Chain of coupled tops:

$$\frac{\hat{\mathbf{S}}_n}{\sqrt{s(s+1)}} \to \vec{R} \in S^2, \qquad \vec{R} = (\cos q\sqrt{1-p^2}, \sin q\sqrt{1-p^2}, p)$$

$$H = J \sum_{n=1}^{N} p_n p_{n+1} + \mu p_n^2 + f(p_n)$$

$p_n \in [-1, 1], q_n \in [0, 2\pi)$ are canonical variables at phase space:



Time evolution:

$$p_n(t) = \text{const.}, \qquad q_n(t) = q_0(0) + tJ(p_{n+1} + p_{n-1} + 2\mu p_n)$$

Periodic orbits/tori

$$q_n(t) = q_n(0) + 2\pi m_n \Longrightarrow$$

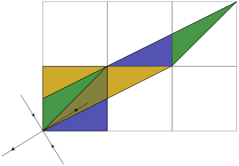
$$2\pi m_n = tJ\left(p_{n+1} + p_{n-1} + 2\mu p_n\right), \qquad m_n \in \mathbb{Z}$$

Particular case $t = t_0 \equiv \pi/J$:

$$p_{n+1} + p_{n-1} + 2\mu p_n = 0 \mod 2, \qquad n = 1, \ldots, N$$

For $2\mu \in \mathbb{Z} \Longrightarrow$

The same equation as for cat map periodic orbits!



$$\begin{pmatrix} q_{n+1} \\ p_{n+1} \end{pmatrix} = \begin{pmatrix} 2\mu - 1 & 1 \\ 2\mu - 2 & 1 \end{pmatrix} \begin{pmatrix} q_n \\ p_n \end{pmatrix} \mod 1,$$

Periodic orbits at $t = t_0$

$$p_{n+1} + p_{n-1} + 2\mu p_n = 0 \mod 2, \qquad n = 1, \dots, N$$

A) For $|\mu| > 1$ full **spatial chaos**. Number of PO grows exponentially with N:

$$\#\mathrm{PO} \sim \Lambda^N, \qquad \Lambda + \Lambda^{-1} = 2\mu, \quad \Lambda > 1$$

B) For $|\mu| < 1$ all PO of the cat map are elliptic: $\Lambda_{1,2} = e^{\pm i\omega}$.
Number of PO orbits does not grow with N!
Non-trivial PO appear if $\omega = 2\pi r/m$, $\mu = \cos(2\pi r/m)$

C) For $\mu = 1$ all PO are parabolic: $\Lambda_{1,2} = 1$.
Number of PO grows only algebraically with N

For (B) and (C) no spatial chaos!

Periodic orbits/tori. Time dependence.

Shorter times $t < t_0$. Dual (spatial) dynamical system is contracting. **Example:** $t = t_0/\ell$ with ℓ integer. PO equation:

$$0 \bmod 2\ell = p_{n+1} + p_{n-1} + 2\mu p_n,$$

Any solution of this equation is also solution of

$$0 \bmod 2 = p_{n+1} + p_{n-1} + 2\mu p_n,$$

$$\Downarrow$$

POs of spin chain at $t = t_0/\ell$ are subset of POs at $t = t_0$.

Longer times $t > t_0$. Dual system is expanding. POs of spin chain at $t = t_0$ are subset of POs at $t = t_0\ell$.

Number of POs grows algebraically $\sim (t/t_0)^N$ **with** t**, but exponentially with** N **if** $|\mu| > 1$.

Quantum dual evolution

$$\underbrace{\mathrm{Tr}\; e^{i\hat{H}t(s+1/2)}}_{\textit{time evolution}} = \sum_{\{s_n=-s,\cdots+s\}} e^{i2tJ/(2s+1)\sum_{n=1}^{N} s_n s_{n+1}+\mu s_n^2+f(s_n)}$$

$$= (2s+1)^{N/2} \quad \underbrace{\mathrm{Tr}\; \tilde{U}^N}_{\textit{dual evolution}}$$

$$\langle m|\tilde{U}|n\rangle = \frac{1}{\sqrt{2s+1}} e^{i2tJ(mn+\mu n^2+f(n))/(2s+1)}, \quad m,n \in [-s,+s]$$

In general $\tilde{U}$ is **non-unitary** $(2s+1)\times(2s+1)$ matrix

Spectral form factor: $K(t,N) = \langle|\mathrm{Tr}\;\tilde{U}^N|^2\rangle$

The roles of t and N are exchanged
M. Akila, P. Braun, D. Waltner, B. G., T. Guhr (2017)

Quantum chaos in integrable system

For $t = t_0$, $\tilde{U}$ is unitary! In particular if $|\mu| > 1$ integer, $\tilde{U}$ is **quantum (perturbed) cat map !** $\Longrightarrow$

RMT can be used to evaluate $\langle |\mathrm{Tr}\, \tilde{U}^N|^2 \rangle$. E.g., for broken time reversal invariance:

$$K(t = t_0, N) = \begin{cases} N, & \text{for } 1 \leq N \leq 2s+1 \\ 2s+1, & \text{for } 2s+1 \leq N \end{cases}$$

Semiclassical point of view. For large chains $N \sim 2s+1$ **correlations between PO** play important role even for short times! Emergence of **spatial Sieber-Richter pairs.**

Quantum chaos for shorter times

For $t < t_0$, $\tilde{U}$ is non-unitary

$$\langle m|\tilde{U}|n\rangle = \frac{1}{\sqrt{2s+1}} e^{i\alpha 2\pi(mn+\mu n^2)/(2s+1)}, \qquad \alpha = t/t_0$$

If $\alpha = 1/\ell$ and ℓ is integer

$$\mathrm{Tr}\,\tilde{U}^N = \ell^{N/2}\,\mathrm{Tr}\,(U_{cat}P)^N\,,$$

where U_{cat} is $\ell(2s+1) \times \ell(2s+1)$ unitary quantization of cat map and P is projection on a part of the phase space $p \in (-\alpha, \alpha) \Longrightarrow$

$\tilde{U}$ is **quantum cat map with absorption**

For $N \to \infty$, $\mathrm{Tr}\,\tilde{U}^N$ is dominated by the largest eigenvalue of $\tilde{U}$

For smaller N non-unitary RMT ensembles can be used to evaluate $K(t, N) = \langle |\mathrm{Tr}\,\tilde{U}^N|^2 \rangle$

Quantum chaos for longer times

For $t > t_0$, $\tilde{U}$ is non-unitary.
In a special case $t = t_0\ell$ and $(2s+1)/\ell$ is an integer

$$\mathrm{Tr}\,\tilde{U}^N = \ell^{N/2}\mathrm{Tr}\,(U_{cat})^N\,,$$

where U_{cat} is $\ell^{-1}(2s+1) \times \ell^{-1}(2s+1)$ unitary quantization of cat map.
The form factor:

$$K(t,N) = \left(\frac{t}{t_0}\right)^N \langle|\mathrm{Tr}\,(U_{cat})^N|^2\rangle$$

Can be evaluated by RMT or semiclassically i.e., by PO correlations

Summary

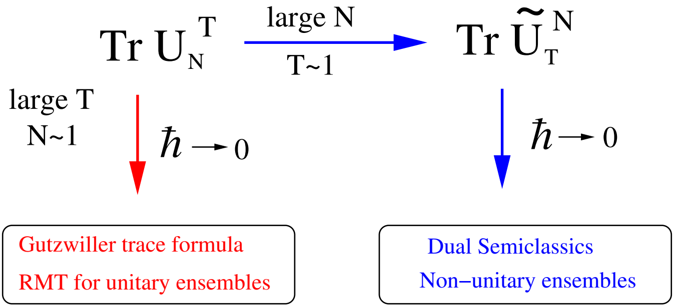
- Number of periodic orbits in integrable systems can grow exponentially with number of particles. **Spatial chaos**

- Periodic orbits do have non-trivial correlations. **Spatial Sieber-Richter pairs.**

- These correlations are important for long range energy (short time) spectral statistics

Beyond integrable systems

Many-body semiclassics:



- ☐ Semiclassics based on PO correlations

M. Akila, D. Waltner, P. Braun, B. G., T. Guhr (2018)

- ☐ Universality for short T. Application of RMT.