Kac–Rice fixed point analysis for large complex systems

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Random Matrices, Integrability, and Complex Systems
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Outline

What? Counting fixed points in large complex systems

Why? Stability analysis of large complex systems

How? Kac–Rice formalism + random matrix techniques

What is a complex system?

Complex systems

Discrete time dynamical systems represent a paradigm in the study of complex and chaotic systems

$$x_{n+1} = f(x_n)$$

- $f : \mathbb{R}^N \to \mathbb{R}^N$ is some “complicated” map
- $x_n \in \mathbb{R}^N$ is a point in space at time $n > 0$
- x_0 is an initial condition

Will a large complex system be stable?

The random linear model

Random linear model (version 1):

$$f(x) = \mathbf{G} \cdot x$$

- $\mathbf{G}$ is an $N \times N$ matrix whose entries are i.i.d. centred Gaussians with variance σ^2/N.

Random linear model (version 2):

$$\mathbb{E}[f(x)] = 0 \qquad \mathbb{E}[f(x) \otimes (f(y))^T] = \frac{\sigma^2}{N}(x \cdot y)\,\mathbf{I}_N$$

where

- $x \cdot y$ is the usual Euclidean inner product
- $\mathbf{I}_N$ is an $N \times N$ identity matrix

The random linear model

Spectral radius

$$\rho(\sigma) = \max\{|\lambda| : \lambda \text{ is a eigenvalue of } \mathbf{G}\}$$

- If $\rho(\sigma) < 1$ then the linear model is stable
- If $\rho(\sigma) > 1$ then the linear models is unstable

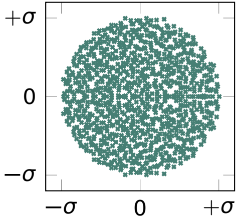
May–Wigner transition:

Spectral radius

$$\lim_{N\to\infty} \rho(\sigma) = \sigma$$

The large-N linear model is

- stable if $\sigma < 1$
- unstable if $\sigma > 1$



1 000 × 1 000 matrix

How to construct a nonlinear model?

Symmetries of the random linear model

The random linear model is defined by

$$\mathbb{E}[f(x)] = 0, \qquad \mathbb{E}[f(x) \otimes (f(y))^T] = \frac{\sigma^2}{N}(x \cdot y)\, \mathbf{I}_N$$

where

- $x \cdot y$ is the usual Euclidean inner product
- $\mathbf{I}_N$ is an $N \times N$ identity matrix

Symmetries:

- domain-isotropic
  $$\mathbb{E}[f(Ux) \otimes (f(Uy))^T] = \mathbb{E}[f(x) \otimes (f(y))^T], \qquad \forall\, U \in O(N)$$
- codomain-isotropic
  $$\mathbb{E}[Vf(x) \otimes (Vf(y))^T] = \mathbb{E}[f(x) \otimes (f(y))^T], \qquad \forall\, V \in O(N)$$

The nonlinear model

Let f be centred Gaussian random map with covariance

$$\mathbb{E}[f(x) \otimes f(y)^T] = \frac{1}{N}\kappa\left(\frac{\|x-y\|^2}{2}\right)\mathbf{I}_N$$

where

- $\|\bullet\|$ is the Euclidean norm
- $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ is a "nice" function

Symmetries:

- domain-isotropy
- codomain-isotropy
- homogeneity

$$\mathbb{E}[f(x+a) \otimes (f(y+a))^T] = \mathbb{E}[f(x) \otimes f(y)^T] \qquad \forall\ a \in \mathbb{R}^N$$

How many fixed point does a complex system have?

Mean number of fixed points

Theorem

Let $\mathcal{N}_f$ be an integer valued random variable, which gives the number of fixed points for the dynamical system described earlier, then we have

$$\mathbb{E}[\mathcal{N}_f] = \mathbb{E}_{\mathbf{G}}[|\det(\sigma \mathbf{G} - \mathbf{I}_N)|]$$

where

- $\mathbf{G}$ is an $N \times N$ matrix whose entries are i.i.d. centred Gaussian random variables with variance $1/N$
- $\mathbb{E}_{\mathbf{G}}$ is the expectation w.r.t. $\mathbf{G}$
- $\mathbf{I}_N$ is the $N \times N$ identity matrix
- $\sigma = (-\kappa'(0))^{1/2} > 0$ is a constant

Sketch of derivation (part I)

Our problem of finding the number of fixed points is equivalent to the multivariate crossing problem

$$0 = f(x) - x$$

Multivariate Kac–Rice formula:

$$\mathbb{E}[\mathcal{N}_f] = \int_{\mathbb{R}^N} \mathbb{E}_{f,\nabla f}\left[\delta^N(f(x) - x)\left|\det_{ij}(\partial_i f_j(x) - \delta_{i,j})\right|\right] dx$$

where

- $\nabla f = (\partial_j f_i)_{ij}$ is a $\mathbb{R}^{N\times N}$-valued Gaussian random map
- $\mathbb{E}_{f,\nabla f}$ is the joint expectation w.r.t. f and ∇f
- δ_{ij} is the Kronecker delta
- $\delta^N(x)$ is a Dirac delta

Sketch of derivation (part II)

We recall that we have the covariance

$$\mathbb{E}[f_i(x)f_j(y)] = \frac{1}{N}\kappa\Big(\frac{\|x-y\|^2}{2}\Big)\delta_{ij}$$

and consequently

$$\mathbb{E}[f_i(x)f_j(x)] = +\frac{1}{N}\kappa(0)\delta_{ij}$$

$$\mathbb{E}[\partial_k f_i(x)f_j(x)] = 0$$

$$\mathbb{E}[\partial_k f_i(x)\partial_\ell f_j(x)] = -\frac{1}{N}\kappa'(0)\delta_{ij}\delta_{k\ell}$$

- The fields $f(x)$ and $\nabla f(x)$ are uncorrelated and therefore independent
- The variance of $f(x)$ and $\nabla f(x)$ does not dependent on the location x

Sketch of derivation (part III)

$$\begin{aligned}
\mathbb{E}[\mathcal{N}_f] &= \int_{\mathbb{R}^N} \mathbb{E}_{f,\nabla f}\Big[\delta^N(f(x)-x)\Big|\det_{ij}(\partial_i f_j(x)-\delta_{i,j})\Big|\Big]dx \\
&= \int_{\mathbb{R}^N} \mathbb{E}_f\Big[\delta^N(f(x)-x)\Big]\mathbb{E}_{\nabla f}\Big[\Big|\det_{ij}(\partial_i f_j(x)-\delta_{i,j})\Big|\Big]dx \\
&= \int_{\mathbb{R}^N} \mathbb{E}_f\Big[\delta^N(f(0)-x)\Big]\mathbb{E}_{\nabla f}\Big[\Big|\det_{ij}(\partial_i f_j(0)-\delta_{i,j})\Big|\Big]dx \\
&= \mathbb{E}_{\mathbf{G}}\Big[\Big|\det(\sigma\mathbf{G}-\mathbf{I}_N)\Big|\Big]
\end{aligned}$$

where $\mathbf{G} = (G_{ki} = \partial_k f_i(0)/\sigma)_{ki}$ is a centred Gaussian random matrix with covariance

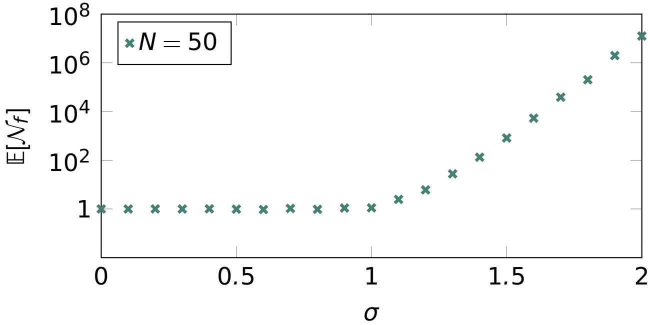
$$\mathbb{E}_{\mathbf{G}}[G_{ki}G_{\ell j}] = \frac{1}{N}\delta_{ij}\delta_{k\ell}$$

How many fixed points does complex system have?

We recall that

$$\mathbb{E}[\mathcal{N}_f] = \mathbb{E}_{\mathbf{G}}\Big[|\det(\sigma\mathbf{G} - \mathbf{I}_N)|\Big]$$

where right-hand side is easy to estimate numerically



Multi-layered systems

Multilayered systems

Let $f_1, \ldots, f_D$ be independent Gaussian maps as described earlier, i.e.

- domain-isotropic
- codomain-isotropic
- homogeneous

Consider a dynamical system $x_{n+1} = f(x_n)$. If

$$f(x) = f_D \circ \cdots \circ f_2 \circ f_1(x),$$

then we say that the system is multilayered with depth D.

If $D = 1$, then we say that the system is single-layered.

Mean number of fixed points

Theorem

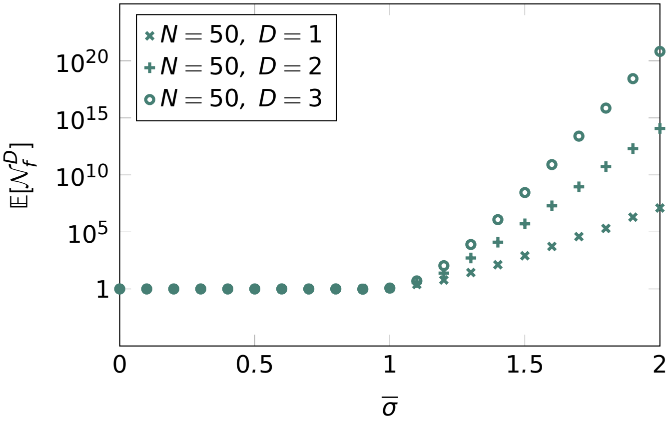
Let $\mathcal{N}_f^D$ be an integer valued random variable, which gives the number of fixed points for the dynamical system described earlier, then we have

$$\mathbb{E}[\mathcal{N}_f^D] = \mathbb{E}_{\mathbf{G}_1,\ldots,\mathbf{G}_D}[|\det(\overline{\sigma}^D \mathbf{G}_1 \cdots \mathbf{G}_D - \mathbf{I}_N)|]$$

where

- $\mathbf{G}_1, \ldots, \mathbf{G}_D$ are independent $N \times N$ matrices whose entries are i.i.d. centred Gaussian random variables with variance $1/N$
- $\mathbb{E}_{\mathbf{G}_1,\ldots,\mathbf{G}_D}$ is the joint expectation w.r.t. $\mathbf{G}_1, \ldots, \mathbf{G}_D$
- $\overline{\sigma} = (\sigma_1 \cdots \sigma_D)^{1/D}$ is the geometric mean
- $\sigma_d = (-\kappa_d'(0))^{1/2} > 0$ is a constant

Number of fixed points for a multilayered system



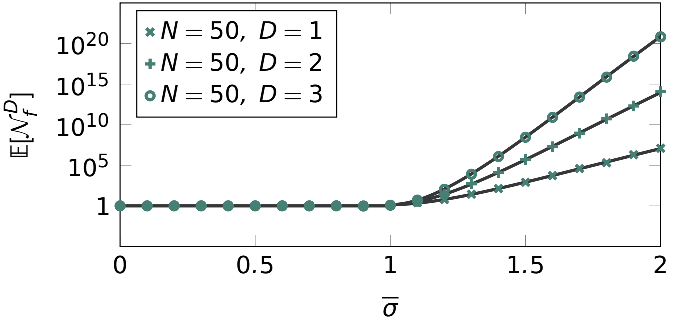
Asymptotic results for large systems

The high dimensional case

We have

$$\mathbb{E}[\mathcal{N}_f^D] \sim \begin{cases} \sqrt{2}e^{ND(\log\overline{\sigma}+\frac{1}{2}(\frac{1}{\overline{\sigma}^2}-1))}, & \overline{\sigma} > 1 \\ 1, & \overline{\sigma} < 1 \end{cases}$$

for large N.



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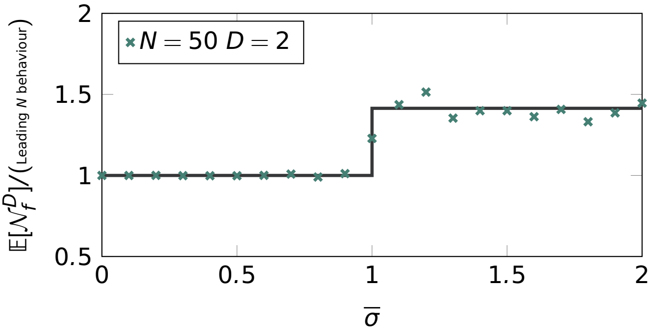


Sketch of derivation of asymptotic result (Part I)

We are interested in the quantity

$$\mathbb{E}_{\mathbf{X}}[|\det(\mathbf{X} - \lambda \mathbf{I}_N)|]$$

where **X** is an asymmetric random matrix.

Assume **X** has n real and $2m$ complex eigenvalues

- $\lambda_1, \ldots, \lambda_n$: real eigenvalues
- $z_1, z_1^*, \ldots, z_m, z_m^*$: complex eigenvalues

and the JPDF for the eigenvalues is

$$P_{N,n}(\lambda_1, \ldots, \lambda_n, z_1, \ldots, z_m) = \frac{1}{n!m!} \frac{1}{Z_N} \prod_{k=1}^{n} w_{\mathbb{R}}(\lambda_k) \prod_{\ell=1}^{m} w_{\mathbb{C}}(z_\ell) \times |\Delta(\lambda_1, \ldots, \lambda_n, z_1, z_1^*, \ldots, z_m, z_m^*)|$$

Sketch of derivation of asymptotic result (Part II)

We are interested in the quantity

$$\mathbb{E}_{\mathbf{X}}[|\det(\mathbf{X} - \lambda \mathbf{I}_N)|]$$

where $\mathbf{X}$ is an asymmetric random matrix.

Lemma

$$\mathbb{E}_{\mathbf{X}}\Big[\,|\det(\mathbf{X} - \lambda \mathbf{I}_N)|\,\Big] = \frac{1}{w_{\mathbb{R}}(\lambda)} \frac{Z_{N+1}}{Z_N} \rho_{\mathbb{R},N+1}(\lambda)$$

where $\rho_{\mathbb{R},N}(\lambda)$ is the mean spectral density of the real eigenvalues of $N \times N$ matrix

The proof the lemma is based on the trivial identity

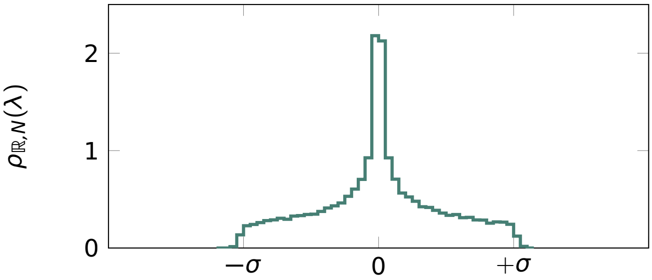
$$\Delta(x_0, x_1, \ldots, x_N) = \Delta(x_1, \ldots, x_N) \prod_{k=1}^{N} (x_k - x_0).$$

Sketch of derivation of asymptotic result (Part III)

We are interested in the quantity

$$\mathbb{E}_{\mathbf{X}}[|\det(\mathbf{X} - \lambda \mathbf{I}_n)|] = \frac{1}{w_{\mathbb{R}}(\lambda)} \frac{Z_{N+1}}{Z_N} \rho_{\mathbb{R},N+1}(\lambda)$$

where **X** is an asymmetric random matrix.



Summary

Summary

Similarly to May's random linear model, our non-linear model has a phase transition at the critical point $\overline{\sigma}_c = 1$.

- If $\overline{\sigma} < 1$ our non-linear system has a single fixed point
- If $\overline{\sigma} > 1$ then the number of fixed points grows exponentially fast with N

The expected number of fixed points is universal in the sense that does not depend on the full structure of the covariance functions

$$\kappa_1, \ldots, \kappa_D : \mathbb{R}_+ \to \mathbb{R}_+$$

but only on local quantity

$$\overline{\sigma} = ((-\kappa_1'(0)) \cdots (-\kappa_D'(0))^{1/D}$$

References

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Thanks for your attention!