Experimental Notes on the Partial Sums of the Riemann Zeta Function

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Partial Sums of Zeta & the Spectral Functions

The Riemann zeta function is defined in the critical strip $0 \leq Re(z) \leq 1$ by

$$\zeta(z) := \frac{1}{1-2^{1-z}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^z}.$$

For $N \in \mathbb{N}$ Let us consider the corresponding partial sums

$$S_N(z) := \frac{1}{1-2^{1-z}} \sum_{k=1}^{N} \frac{(-1)^{k+1}}{k^z}.$$

Figure 1 shows the typical behavior

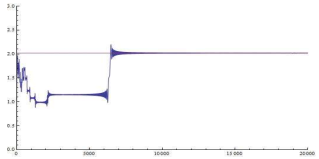


Figure 1: Values of $|S_N(0.5(1-e^{-2}) + 2 \cdot 10^4 i)|$ for $N = 1, \ldots, 2 \cdot 10^4$.

Define $N_m := \left[\frac{Im(z)}{(2m+1)\pi}\right]$. The partial sums preform steep surges around N_m (ass seen in Fig. 1) This leads us the define the **m-th spectral function** to be

$$\lambda_m(z) := \zeta(z) - S_{\frac{1}{2}(N_{m+1}-N_m)} = \frac{1}{1-2^{1-z}} \sum_{n=\left[\frac{Im(z)}{2m\pi}\right]}^{\infty} \frac{(-1)^{k+1}}{k^z}.$$

The Euler-Sondow Formula and Spectrum

The Euler-Sondow formula expresses zeta in the complex plane as the following series (see [2]):

$$\zeta(z) := \frac{1}{1-2^{1-z}} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+1)^z}.$$

We similarly define the partial sums of the Euler-Sondow formula:

$$\tilde{S}_N(z) := \frac{1}{1-2^{1-z}} \sum_{n=0}^{N} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+1)^z}.$$

Figure 2 shows a comparison between the classical $S_N(z)$ and the ES-partial sums $\tilde{S}_N(z)$:

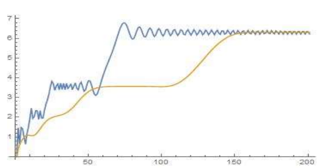


Figure 2: $|S_N(z)|$ and $|\tilde{S}_N(z)|$ for $z = 0.05 + 200i$ and $N = 0, \ldots, 200$

We similarly define the corresponding **spectral functions associated to the Euler-Sondow formula**

$$\tilde{\lambda}_m(z) := \frac{1}{1-2^{1-z}} \sum_{n=\left[\frac{Im(z)}{m\pi}\right]}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(k+1)^z}.$$

The First Spectral Function and the Core

In the spectral theory of differential operators the first eigenvalue is known to have a distinguished role (fundamental tone, ground state, vacuum, etc…). Figure 3 shows the regulated behavior of the first spectral function $\tilde{\lambda}_1(z)$ compared to zeta.

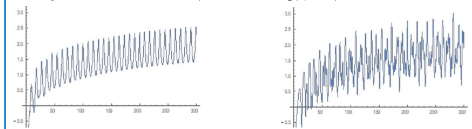


Figure 3: $\log(\tilde{\lambda}_1(0.05 + yi))$ and $\log|\zeta(0.05 + yi)|$ for $0 \leq y \leq 200$.

Let us define the function

$$Core(z) := \frac{2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right)\Gamma(1-z)}{2^{z-1}-1}.$$

Figure 4 shows a comparison between $Core(z)$ and $\tilde{\lambda}_1(z)$

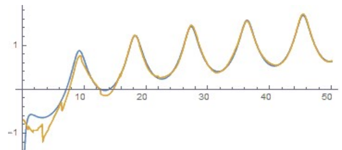


Figure 4: $\log|Core(yi)|$ and $\log|\lambda_1(yi)|$ for $0 \leq y \leq 50$

Empirical verification shows that $Core(z)$ is a rather good approximation of the first spectral function $\tilde{\lambda}_1(z)$. In fact, we have:

Conjecture: For $0 \leq Re(z) \leq \frac{1}{2}$ we have

$$\left|\log\left(\tilde{\lambda}_1(z)\right) - \log(Core(z))\right| \leq e^{-\frac{Im(z)}{20}-1}.$$

The Higher Spectral Functions

Figure 5 shows a comparison between the first and second spectral function

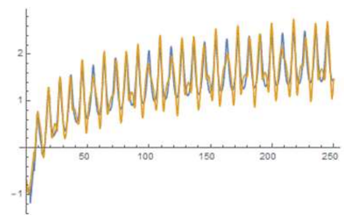


Figure 5: $\log|\tilde{\lambda}_1(z)|$ and $\log|\tilde{\lambda}_2(z)|$ for $0 < y < 250$:

Motivated by spectral theory of differential operators we **define the j-th relative spectral function** to be

$$\delta_j(z) := Log\left(\frac{\tilde{\lambda}_j(z)}{\tilde{\lambda}_{j+1}(z)}\right).$$

Figure 6 shows an illustration of, $\delta_1(z)$, the first relative spectral function:

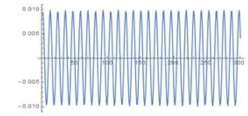


Figure 6: $\delta_1(0.5(1-e^{-2}) + yi)$ for $300 \leq y \leq 450$.

In particular, $\delta_1(z)$ is seen to have a strong periodic signal of period $\frac{2\pi}{\log(3)}$. Figure 7 shows the spectral Fourier decomposition of δ_j for $j = 1,2,3$.

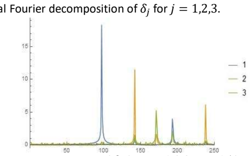


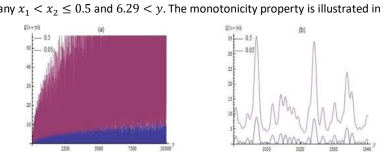
Figure 7: Fourier decomposition of δ_j for $j = 1,2,3$ with $150 \leq Im(z) \leq 650$.

Relations To Zeta Monotonicity

In [3] R. Spira showed that the Riemann Hypothesis is equivalent to the following monotonicity property

$$|\zeta(x_2 + yi)| < |\zeta(x_1 + yi)|$$

for any $x_1 < x_2 \leq 0.5$ and $6.29 < y$. The monotonicity property is illustrated in Fig. 8



domain $0 < y < 10000$ (a) and $1000 < y < 1040$ (b).

Let us define the function

$$\eta(y,t) := e^t(|\zeta(0.5(1-e^{-t}) + yi)| - |\zeta(0.5 + yi)|)$$

Note that the monotonicity conjecture implies that $e^{-t_2}\eta(y,t_2) < e^{-t_1}\eta(y,t_1)$ for any $t_1 < t_2$ and $6.29 < y$. Fig. 9 Illustrates η-monotonicity for initial values:

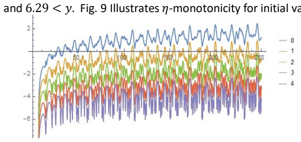


Figure 9: $\log(\eta(y,t)) - t$ for $t = 0, \ldots, 4$ in the domain $6.29 < y < 250$.

Set $\lambda_j^{\eta}(y,t) := e^t\left(|\lambda_j(0.5(1-e^{-t}) + yi)| - |\lambda_j(0.5 + yi)|\right)$. We have

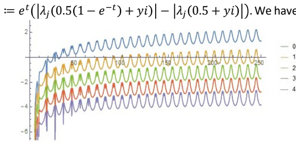


Figure 10: $\log\left(\lambda_1^{\eta}(y,t)\right) - t$ for $t = 0, \ldots, 4$ in the domain $6.29 < y < 250$.

For all (y,t) such that $|\zeta(0.5 + yi)| \neq 0$ we refer to

$$\tilde{\eta}(y,t) := \frac{\eta(y,t)}{|\zeta(0.5 + yi)|} = e^t\left(\frac{|\zeta(0.5(1-e^{-t}) + yi)|}{|\zeta(0.5 + yi)|} - 1\right)$$

as **the taming function of $\eta(y,t)$.** Figure 11 shows the regulated behavior of $\tilde{\eta}(y,t)$:

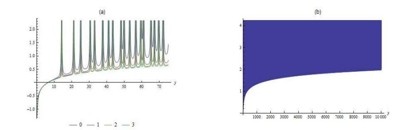


Figure 11: $\tilde{\eta}(y,t)$ for $t = 0,1,2,3$ over the domain $0 < y < 75$ (left) and $\tilde{\eta}(y,3)$ over the domain $0 < y < 10000$ (right).

Consider the function

$$X(y,t) := e^t\left(\frac{|\chi(0.5(1-e^{-t}) + yi)|}{|\chi(0.5 + yi)|} - 1\right) = e^t(|\chi(0.5(1-e^{-t}) + yi)| - 1)$$

Figure 12 illustrates the asymptotic relation between $\tilde{\eta}(y,t)$ and $X(y,t)$:

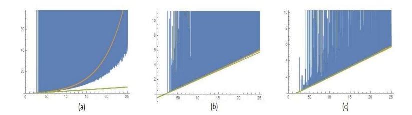


Figure 12: $\tilde{\eta}(e^s,t)$, $X(e^s,t)$, $\frac{1}{4}s + \frac{1}{2}$ for $t = 1$ (a), $t = 5$ (b), $t = 7$ (c) and $s \in [0,25]$

The function $\tilde{\eta}(y,t)$ is seen to have slower rate of growth than $X(y,t)$ (the difference is precisely due to the contribution of the higher spectral function). However, the function $\frac{1}{4}\log(y) + \frac{1}{2}$ is seen to be a bound for both $\tilde{\eta}(y,t)$ and $X(y,t)$. This leads to:

Conjecture: $\frac{1}{4}\log(y) + C \leq \tilde{\eta}(y,t)$ for any $y > 6.29$ and $t \geq 0$ for some $C \geq \frac{1}{2}$.

Let us define the spectral functions of the taming function $\tilde{\eta}(y,t)$ as follows:

$$\lambda_j^{\tilde{\eta}}(y,t) := e^t\left(\frac{|\lambda_j(0.5(1-e^{-t}) + yi)|}{|\lambda_j(0.5 + yi)|} - 1\right)$$

Consider $\tilde{\eta}_\alpha(y,t) := \lambda_{N(\alpha,y)}^{\tilde{\eta}}(y,t)$, where $N(\alpha,y) := 1 + \left[\frac{0.001\alpha y}{\pi}\right]$.

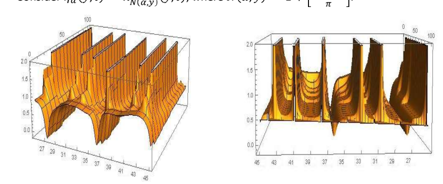


Figure 13: $\tilde{\eta}_\alpha(y,20)$ for $25 < y < 45$ and $0 < \alpha < 100$.

Concluding Remarks

Based on the special properties of the partial sums, we introduced the sequence of spectral functions $\tilde{\lambda}_j(z)$ associated to the Riemann zeta function $\zeta(z)$. We suggest that the spectral functions interpolate between the highly regulated core function $Core(z) \approx \tilde{\lambda}_1(z)$ and $\zeta(z)$ by adding the higher relative spectral functions $\delta_1(z)$ (which are in turn governed by periodic signals of specific periods). Possible relations to the monotonicity conjecture were presented.

References

1. Y. Jerby, *An experimental study of the monotonicity property of the Riemann zeta function*, arXiv:1707.01754.
2. J. Sondow, *Analytic Continuation of Riemann's Zeta Function and Values at Negative Integers Via Euler's Transformation of Series, Proceedings of the American Mathematical Society* Vol. 120, No. 2 (Feb., 1994), 421-424.
3. R. Spira. *Zeros of $\zeta'(s)$ and the Riemann hypothesis.* Illinois J. Math. 17 (1973), no. 1, 147-152.