The Euler-Sondow Formula and Spectrum

Partial Sums of Zeta & the Spectral Functions

The Riemann zeta function is defined in the critical strip $0 \leq \Re(s) \leq 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$ 

For $N \in \mathbb{N}$ let us consider the corresponding partial sums

$$S_N(s) = \sum_{n=1}^{N} \frac{1}{n^s}.$$ 

Figure 1 shows the typical behavior.

The First Spectral Function and the Core

In the spectral theory of differential operators the first eigenvalue is known to have a distinguished role (fundamental tone, ground state, vacuum, etc...). Figure 3 shows the regulated behavior of the first spectral function $\lambda_1(s)$ compared to zeta.

For $N \in \mathbb{N}$, we similarly define the corresponding $N$-th spectral function

$$\lambda_N(s) = \sum_{n=1}^{N} \frac{1}{n^s}.$$

Values of $\lambda_N(s)$ are seen to have a strong periodic signal of period $\log_{10}(N).$ Figure 7 shows the spectral fourier decomposition of $\lambda(s)$ for $s = 1.23$.

Relations To Zeta Monotonicity

In [3] H. S. M. Coxeter showed that the Riemann Hypothesis is equivalent to the following monotonicity property:

$$\frac{d}{ds} \log \zeta(s) < 0 \quad \text{for} \quad \Re(s) \leq 1 \quad \text{and} \quad 0 < \Re(s) < 1 + \varepsilon$$

for any $\varepsilon > 0.$ The monotonicity property is illustrated in Fig. 8.

Figure 12 shows an illustration of $(\lambda(s))$ for $s = 1.23$ with $0 < \Re(s) < 1 + \varepsilon$.

Experimental Notes on the Partial Sums of the Riemann Zeta Function

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