Moments of Moments

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Moments of Characteristic Polynomials

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Characteristic polynomial of an $N \times N$ unitary matrix A on the unit circle in the complex plane:

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Moments:

$$M_N(\beta) = \mathbb{E}_{A \in U(N)} |P_N(A, \theta)|^{2\beta} = \mathbb{E}_{A \in U(N)} \exp(2\beta \log |P_N(A, \theta)|)$$

Calculating Moments

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For $\beta \in \mathbb{N}$

$$M_N(\beta) = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+2\beta)}{(\Gamma(j+\beta))^2} = \prod_{0 \le i,j \le \beta-1} \left(1 + \frac{N}{i+j+1}\right)$$

$$\sim N^{\beta^2} \prod_{m=0}^{\beta-1} \frac{m!}{(m+\beta)!}$$

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$$\mathrm{MoM}_N(k,\beta) := \mathbb{E}_{A\in U(N)}\left(\frac{1}{2\pi}\int_0^{2\pi}|P_N(A,\theta)|^{2\beta}d\theta\right)^k$$

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- Applications to modelling the moments of the Riemann zeta-function in short intervals
- Applications in Quantum Chaos etc

Conjecture

Conjecture (Fyodorov & Bouchaud, Fyodorov & Keating)

When $N \to \infty$

$$\mathrm{MoM}_N(k,\beta) \sim \begin{cases} \left(\frac{(G(1+\beta))^2}{G(1+2\beta)\Gamma(1-\beta^2)}\right)^k \Gamma(1-k\beta^2) N^{k\beta^2} & k < 1/\beta^2 \\ c(k,\beta) N^{k^2\beta^2+1-k} & k \geq 1/\beta^2 \end{cases}$$

where $G(s)$ is the Barnes G-function and $c(k,\beta)$ is a complicated (unspecified) function of k and β.

Heuristic Justification

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For $k \in \mathbb{N}$

$$\mathrm{MoM}_N(k,\beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbb{E} \prod_{j=1}^{k} |P_N(A,\theta_j)|^{2\beta} d\theta_j. \tag{1}$$

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This expression diverges as k approaches $1/\beta^2$ when singularities associated with coalescences of the θ_js become important.

Developing a precise asymptotic in the range $k \geq 1/\beta^2$ requires a Fisher-Hartwig formula that is valid uniformly through coalescences.

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- Webb proved results consistent with the conjecture when $k\beta^2$ is small.
- In the analogous problem in which $\log |P_N(A, \theta)|$ is replaced by a random Fourier series with the same correlation structure, the analogue of conjecture due to Fyodorov and Bouchaud has recently been proved in the regime $k < 1/\beta^2$ for all k and β by Remy using ideas from conformal field theory.

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Let $k, \beta \in \mathbb{N}$. Then

$$\mathrm{MoM}_N(k,\beta) = \gamma_{k,\beta} N^{k^2\beta^2-k+1} + O(N^{k^2\beta^2-k}),$$

where $\gamma_{k,\beta}$ is a polynomial in k and β.

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Recall that for $k \in \mathbb{N}$

$$\mathrm{MoM}_N(k,\beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} \mathbb{E} \prod_{j=1}^{k} |P_N(A,\theta_j)|^{2\beta} d\theta_j.$$

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Instead of evaluating $\mathbb{E} \prod_{j=1}^{k} |P_N(A,\theta_j)|^{2\beta}$ asymptotically, use instead exact representations. These take the form of

- sums associated with partitions coming from representation theory
- multiple integrals

Outline of Proof of First Theorem

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Uses Symmetric Function Theory:

Proposition (Bump & Gamburd)

$$\mathbb{E}_{A\in U(N)}\left(\prod_{j=1}^{k}|P_N(A,\theta_j)|^{2\beta}\right)=\frac{s_{\langle N^{k\beta}\rangle}(e^{i\underline{\theta}})}{\prod_{j=1}^{k}e^{iN\beta\theta_j}},$$

where $s_\lambda(x_1,\ldots,x_n)$ is the Schur polynomial in n variables with respect to the partition λ, where $\langle\lambda^n\rangle=(\overbrace{\lambda,\ldots,\lambda}^{n})$, and

$$e^{i\underline{\theta}}=(\overbrace{e^{i\theta_1},\ldots,e^{i\theta_1}}^{\beta},\ldots,\overbrace{e^{i\theta_k},\ldots,e^{i\theta_k}}^{\beta},\overbrace{e^{i\theta_1},\ldots,e^{i\theta_1}}^{\beta},\ldots,\overbrace{e^{i\theta_k},\ldots,e^{i\theta_k}}^{\beta}).$$

Now

$$s_{\langle N^{k\beta}\rangle}(e^{i\underline{\theta}}) = \sum_T e^{i\theta_1\tau_1}\cdots e^{i\theta_k\tau_k},$$

where the sum is over all semistandard Young tableaux (SSYT) of rectangular shape with $k\beta$ rows by N columns, and

$$\tau_j = t_{2(j-1)\beta+1} + \cdots + t_{2j\beta} \quad j \in \{1,\ldots,k\}.$$

Hence

$$\begin{aligned}\mathrm{MoM}_N(k,\beta) &= \frac{1}{(2\pi)^k}\int_0^{2\pi}\cdots\int_0^{2\pi}\sum_T e^{i\theta_1(\tau_1-N\beta)}\cdots e^{i\theta_k(\tau_k-N\beta)}\prod_{j=1}^k d\theta_j \\ &= \sum_{\tilde{T}} 1,\end{aligned}$$

where the sum is now over $\tilde{T}$, restricted SSYT having $N\beta$ entries from each of the sets

$$\{2\beta(j-1)+1,\ldots,2j\beta\}, \quad j \in \{1,\ldots,k\}.$$

We can now use a well-known relation between the number of SSYT of shape λ with entries in $1, 2, \ldots, n$ and the Schur polynomial $s_\lambda(1, \ldots, 1)$ (if necessary λ is extended with zeros until it has length n), together with the formula

$$s_\lambda(1, 1, \ldots, 1) = \prod_{1 \le i < j \le n} \frac{\lambda_i - \lambda_j + j - i}{j - i},$$

which is a polynomial in $\lambda_i - \lambda_j$.

Since the set of RSSYT is a proper subset of all SSYT, we have that the number of RSSYT of rectangular shape $\lambda = \langle N^{k\beta} \rangle$ is a polynomial in N of degree less than $k^2\beta^2$.

Outline of Proof of Second Theorem

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Define

$$I_{k,\beta}(\theta_1, \dots, \theta_k) = \mathbb{E}_{A\in\ U(N)} \left(\prod_{j=1}^{k} |P_N(A, \theta_j)|^{2\beta} \right),$$

so that

$$\mathrm{MoM}_N(k, \beta) = \frac{1}{(2\pi)^k} \int_0^{2\pi} \cdots \int_0^{2\pi} I_{k,\beta}(\theta_1, \dots, \theta_k) d\theta_1 \cdots d\theta_k.$$

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We then use

$$I_{k,\beta}(\underline{\theta}) = \frac{(-1)^{k\beta}e^{-i\beta N\sum_{j=1}^{k}\theta_j}}{(2\pi i)^{2k\beta}((k\beta)!)^2}\oint\cdots\oint \frac{e^{-N(z_{k\beta+1}+\cdots+z_{2k\beta})}\Delta(z_1,\ldots,z_{2k\beta})^2 dz_1\cdots dz_{2k\beta}}{\prod_{m\leq k\beta<n}(1-e^{z_n-z_m})\prod_{m=1}^{2k\beta}\prod_{n=1}^{k}(z_m+i\theta_n)^{2\beta}}.$$

We deform each of the $2k\beta$ contours so that any one consists of a sum of k small circles surrounding each of the poles at $-i\theta_1, \ldots, -i\theta_k$, given by $\Gamma_{-i\theta_n}$ for $n \in \{1, \ldots, k\}$, and connecting straight lines whose contributions will cancel. This leads to a sum of $k^{2k\beta}$ multiple integrals,

$$I_{k,\beta}(\underline{\theta}) = \frac{(-1)^{k\beta} e^{-i\beta N \sum_{j=1}^{k} \theta_j}}{(2\pi i)^{2k\beta}((k\beta)!)^2} \sum_{\varepsilon_j \in \{1,\ldots,k\}} J_{k,\beta}(\underline{z}; \underline{\theta}; \varepsilon_1, \ldots, \varepsilon_{2k\beta}).$$

where

$$J_{k,\beta}(\underline{z}; \underline{\theta}; \varepsilon_1, \ldots, \varepsilon_{2k\beta}) = \int_{\Gamma_{-i\theta_{\varepsilon_1}}} \cdots \int_{\Gamma_{-i\theta_{\varepsilon_{2k\beta}}}} \frac{e^{-N(z_{k\beta+1}+\cdots+z_{2k\beta})} \Delta(z_1, \ldots, z_{2k\beta})^2 dz_1 \cdots dz_{2k\beta}}{\prod_{m \le k\beta < n} (1 - e^{z_n - z_m}) \prod_{m=1}^{2k\beta} \prod_{n=1}^{k} (z_m + i\theta_n)^{2\beta}}.$$

Next we perform the change of variables,

$$z_n = \frac{v_n}{N} - i\alpha_n,$$

shifting all the contours to be small circles surrounding the origin, and then compute the resulting integrals asymptotically as $N \to \infty$. This gives (after a lengthy calculation)

$$\mathrm{MoM}_N(k,\beta) \sim \gamma_{k,\beta} N^{k^2\beta^2 - k + 1}$$

where

$$\gamma_{k,\beta} = \sum_{\substack{l_1,\ldots,l_{k-1}=0 \\ (\dagger)}}^{2\beta} c_{\underline{l}}(k,\beta)\Big((k-1)\beta - \sum_{j=1}^{k-1} l_j\Big)^{|B_{k,\beta;\underline{l}}|-k+1} P_{k,\beta}(l_1,\ldots,l_{k-1}),$$

with

$$P_{k,\beta}(l_1,\dots,l_{k-1}) = \frac{(-1)^{\sum_{\sigma<\tau}|S^-_{\sigma<\tau}|}}{(2\pi i)^{2k\beta}((k\beta)!)^2}$$

$$\times \int_{\Gamma_0}\cdots\int_{\Gamma_0} \frac{e^{-\sum_{m=k\beta+1}^{2k\beta} v_m}\prod_{\substack{m<n\\ \alpha_m=\alpha_n}}(v_n-v_m)^2}{\prod_{\substack{m\leq k\beta<n\\ \alpha_n=\alpha_m}}(v_n-v_m)\prod_{m=1}^{2k\beta} v_m^{2\beta}}$$

$$\times \Psi_{k,\beta;\underline{l}}(((k-1)\beta-\sum_{j=1}^{k-1} l_j)\underline{v})\prod_{m=1}^{2k\beta} dv_m.$$

We have therefore proved the conjecture provided that we can show that $\gamma_{k,\beta}\neq 0$. A (lengthy) calculation shows this to be the case, and that in fact $\gamma_{k,\beta}>0$.

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We have therefore proved the conjecture provided that we can show that $\gamma_{k,\beta}\neq 0$. A (lengthy) calculation shows this to be the case, and that in fact $\gamma_{k,\beta}>0$. . Moreover, it follows from the general formula for $\gamma_{k,\beta}$ that it is a polynomial in k and β.

The contour integrals analysed asymptotically above can also be evaluated exactly, using the residue theorem, yielding explicit formulae for the polynomials, For example

$$\mathrm{MoM}_N(1,1) = N+1$$

$$\mathrm{MoM}_N(2,1) = \frac{1}{6}(N+3)(N+2)(N+1)$$

$$\mathrm{MoM}_N(3,1) = \frac{1}{2520}(N+5)(N+4)(N+3)(N+2)(N+1)(N^2+6N+21)$$

$$\begin{aligned}\mathrm{MoM}_N(4,1) = {} & \frac{1}{194594400}(N+7)(N+6)(N+5)(N+4)(N+3) \\ & \times (N+2)(N+1)(7N^6+168N^5+1804N^4+10944N^3 \\ & +41893N^2+99624N+154440)\end{aligned}$$

$$\mathrm{MoM}_N(1,2) = \frac{1}{12}(N+1)(N+2)^2(N+3)$$

$$
\begin{aligned}
\mathrm{MoM}_N(2,2) = \frac{1}{163459296000}&(N+7)(N+6)(N+5)(N+4)(N+3)\\
&\times (N+2)(N+1)(298N^8 + 9536N^7 + 134071N^6 + 1081640N^5\\
&+ 549437N^4 + 18102224N^3 + 38466354N^2\\
&+ 50225040N + 32432400)
\end{aligned}
$$

etc.

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- The integer (k) moments of the integer (β) moments are polynomials in N of degree $k^2\beta^2 - k + 1$.
- This fits with the general conjecture.
- We recover a formula for the leading coefficient in this case.
- The proof combines both the representation-theoretic and analytic approaches.
- The polynomials in question can be computed explicitly.