Universal random matrix kernels from quantum mechanical hydrogen atom problem

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Tribute

- John Wishart (∗1898, Montrose, †1956, Acapulco)
  Biometrika **20A** (1928) 32
- Erwin Schroedinger (∗1887, Vienna, †1961, Vienna)
  Ann. Physik **79** (1926) 361
- Salomon Bochner (∗1899, Kraków, †1982, Houston)
  Math. Z. **29** (1929) 730.

χ^2 distribution and Wishart ensemble

- We consider x_i from iid standard Gaussian distribution and we form $y = \sum_{i=1}^{T} x_i^2$. Pdf of such distribution reads
  $p(y) \sim y^{T/2-1} e^{-y/2}$
  Crucial distribution when analyzing variance, testing hypothesis etc..
- We consider vectors $\vec{x}_i$ from standard real/complex Gaussian distributions and we form matrix X

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1T} \\ \vdots & \vdots & \vdots \\ x_{N1} & \cdots & x_{NT} \end{pmatrix}$$

Then we form correlation matrix $M = \frac{1}{T} X X^{\dagger}$.
Wishart distribution (for the complex case and $N \leqslant T$) reads
$P(M) \sim \det M^{T-N} e^{-T \operatorname{tr} M}$

Switching to spectra

- $P_N(\lambda_1, ..., \lambda_N) \sim \prod \lambda_i^{T-N} e^{-T \sum \lambda_i} \Delta(\Lambda)^2$
- **Slater determinant**

$$P_N(\lambda_1, ..., \lambda_N) = \frac{1}{N!} \left[\det \psi_{j-1}^{(N)}(\lambda^k)|_{j,k=1}^N\right]^2 = \frac{1}{N!} \left[\det K_N(\lambda_i, \lambda_j)\right]$$

with the **kernel**

$$K_N(\lambda, \mu) = \sum_{l=0}^{N-1} \psi_l^{(N)}(\lambda) \psi_l^{(N)}(\mu)$$

Here $\psi_l^{(N)}(\lambda) = e^{-T\lambda/2} \lambda^{(T-N)/2} P_l^{(N)}(\lambda)$ **is a wave function**

- **Quantum Mechanics I**

  Radial Schroedinger eq. for hydrogen atom (in units $2\mu = 1$).
  Completely integrable system for any N, T!

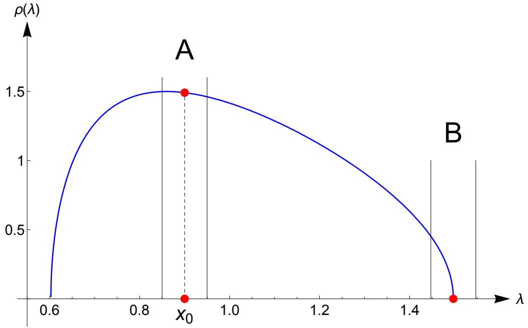
My favorite citation...

"Old" quantum theory (1914-1917)

- $\hbar \sim \frac{1}{T} \to 0$
- Bohr-Sommerfeld formula $\oint p(r)dr = (n + \frac{1}{2})2\pi\hbar$
- Semi-classically, $p^2 - \frac{1}{r} + \frac{l(l+1)}{r^2} = E$.
  In the limit $N, T \to \infty$, $N/T = c$ fixed, where N, T are related to n, l, Bohr-Sommerfeld formula **is** Marchenko-Pastur formula for Wishart ensemble (new result?)
- $$\int_{r_-}^{r_+} \rho(x)dx = 1 \quad \text{with} \quad \rho(x) = \frac{1}{2\pi c x}\sqrt{(r_+ - x)(x - r_-)}$$

  where $r_\pm = (1 \pm \sqrt{c})^2$ are classical turning points.
- Note that same reasoning converts the harmonic oscillator ellipse $E = p^2 + \frac{x^2}{4}$ into Wigner semi-ellipse $\rho(x) = \frac{1}{2\pi}\sqrt{4 - x^2}$ [T. Tao].

Quantum microscopy



Quantum microscopy, cont.

- "Bulk"

$$n_{bulk} \sim N \int_{x_0 - s/2}^{x_0 + s/2} \rho(x) dx \sim Ns\rho(x_0)$$

so we have scaling $s \sim 1/(N\rho(x_0))$

- "Soft edge"

$$n_{soft} \sim N \int_{-s/2}^{s/2} \sqrt{x} dx \sim Ns^{3/2}$$

so we have scaling $s \sim 1/N^{2/3}$.

- "Hard edge"

$$n_{hard} \sim N \int_0^s \frac{dx}{\sqrt{x}} \sim N\sqrt{s}$$

so we have scaling $s \sim 1/N^2$.

Spectral deformation of the QM projection operator

- **Quantum mechanics II**
  $\hat{K}_N = \sum_i^N |\psi_i >< \psi_i|$ is a projection operator ($\hat{K}_N^2 = \hat{K}_N$)
- Spectral bound $\hat{H}_N \leqslant E_N$, with $E_N = -1/4N^2$, combined with pertinent microscopic scaling, shows the deformation of the domain of the operator $\hat{H}$. Deciphering this deformation yields a microscopic form of the kernel K for each pertinent scaling, respectively.

Bulk

Bound $\hat{H}_N \leqslant E_N$, or explicitly

$$\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} + \frac{1+2k+\nu}{2x} - \frac{\nu^2}{4x^2} \geqslant \frac{1}{4}$$

with the scaling $\frac{x}{T} = x_0 + \frac{s}{N\rho(x_0)}$, converts in the large N limit ($k \sim N$, $\nu = T - N$) onto $\frac{d^2}{ds^2} \geqslant \frac{(x_0-r_+)(x_0-r_-)}{4c^2x^2\rho^2(x_0)}$, therefore $-\frac{d^2}{ds^2} \leqslant \pi^2$

- QM suggests the use of plane waves, then $(2\pi t)^2 \leqslant \pi^2$, so the deformation is the limitation of all possible momenta t to the strip $[-1/2, 1/2]$.
- Identity operator $\mathbf{1}_{tt'} = \delta(t-t')$ (completeness)
  $F(t') = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{2\pi i t' s} e^{-2\pi i t s} ds\right] F(t)dt$ gets deformed to projecion operator
  $\mathbf{P}[F(t')] = \int_{-\infty}^{\infty}\left[\int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i t' s} e^{-2\pi i t s} ds\right] F(t)dt$
- Hence the universal Dyson kernel

$$\delta(t-t') \rightarrow K_{Sine}(t,t') = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i t' s} e^{-2\pi i t s} ds = \frac{\sin(\pi(t'-t))}{\pi(t'-t)}$$

Soft edge

We repeat similar reasoning for soft edge.

- Deformation in the case of soft edge converts the Schroedinger eq. in the large N limit onto the bound $-\frac{d^2}{ds^2} + s \leqslant 0$ (triangular potential). Role of Fourier transforms is played by the pair of Airy transforms

$$F(t) = A[f(z)] = \int_{-\infty}^{\infty} Ai(t-z)f(z)dz$$

and its inverse

$$f(z) = \int_{-\infty}^{\infty} F(t)Ai(t-z)dt.$$

- This transform leads to the spectral condition

$$t \leqslant 0$$

Soft edge cont.

- Combining both Airy transforms we obtain the identity operator

$$F(t') = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} Ai(t' - z)Ai(t - z)dz \right] F(t)dt$$

- The deformation condition projects the above identity operator onto

$$\mathbf{P}[F(t')] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{0} Ai(t' - z)Ai(t - z)dz \right] F(t)dt$$

so the kernel, understood as a projection, reads

$$K_{Airy}(t, t') = \int_{-\infty}^{0} Ai(t' - z)Ai(t - z)dz = \frac{Ai(t')Ai'(t) - Ai'(t')Ai(t)}{t' - t}$$

where on the r.h.s. we presented the more familiar form of the Airy kernel based on relation

$$\frac{d}{dz} \left[\frac{Ai(t' - z)Ai'(t - z) - Ai'(t' - z)Ai(t - z)}{t' - t} \right] = Ai(t' - z)Ai(t - z)$$

Hard edge

We repeat similar reasoning for the hard edge.

- Deformation in the case of hard edge yields the bound

$$\Delta_\nu \equiv -\frac{d^2}{dz^2} - \frac{1}{z}\frac{d}{dz} - \frac{\nu^2}{z^2} \leqslant 1$$

  where on the l.h.s. we recognize Bessel operator, appearing in quantum mechanical problems with polar angle symmetry and $\nu = T - N \sim O(1)$.
- To see the deformation caused by hard edge scaling in the above equation we define *Hankel transform*

$$F_\nu(t) = H_\nu[f(z)] = \int_0^\infty zf(z)J_\nu(z)dz$$

  and the inverse Hankel transform is given as

$$f(z) = \int_0^\infty tF_\nu(t)J_\nu(tz)$$

  Since the Hankel transform of the Bessel operator reads $H_\nu[\Delta_\nu f(z)] = t^2F_\nu(t)$, the spectral deformation in dual variable t (note that t cannot be negative) reads simply $0 \leqslant t \leqslant 1$

Hard edge - cont.

- Combining both Hankel transforms we obtain (modulo change of the variables) the identity operator

$$F_\nu(t') = \int_0^\infty \left[\int_0^\infty z t J_\nu(t' z) J_\nu(tz) dz \right] F_\nu(t) dt$$

The deformation condition projects the above identity operator onto

$$\mathbf{P}[F_\nu(t')] = \int_0^\infty \left[\int_0^1 z t J_\nu(t' z) J_\nu(tz) dz \right] F_\nu(t) dt$$

so the kernel, understood as a projection, reads

$$K_{Bessel}(t, t') = \int_0^1 z t J_\nu(t' z) J_\nu(tz) dz$$

- Using Lommel integral we arrive at the more familiar form

$$K_{Bessel}(x, y) = \frac{J_\nu(\sqrt{x}) J'_\nu(\sqrt{y}) \sqrt{y} - \sqrt{x} J'_\nu(\sqrt{x}) J_\nu(\sqrt{y})}{2(x - y)}$$

No-go theorem

- **Bochner theorem**
  If an infinite sequence of polynomials $P_n(x)$ satisfies a second order eigenvalue eq.

  $$p(x)P_n^{''} + q(x)P_n^{'} + r(x)P_n = \lambda_n P_n$$

  then $p(x), q(x), r(x)$ must be polynomials of degree 2,1, and 0, respectively
- If additionally polynomials are orthogonal, the only solutions are polynomials of Jacobi, Laguerre or Hermite
- This leads to universal limit of determinantal processes for Sturm-Louiville operators [Bornemann, 2016], i.e. for the GUE, LUE and JUE (a.k.a. MANOVA) - yielding sine, Airy and Bessel $\beta = 2$ universality.

How to go out from the No-go theorem

- Consider higher order equation comparing to Sturm-Liouville (S-L) problem, e.g. third order diff. equation.
- This leads to nonhermitian operator $\mathcal{H}$.
- Eigenvalue problem is more complicated

$$\mathcal{H}|R_n >= \lambda_n|R_n > \quad < L_n|\mathcal{H}^\dagger = \lambda_n < L_n|$$

where $|R_n >$ and $|L_n >$ are right and left eigenvectors to λ_n.
- **Quantum mechanics III (nonhermitian)**:
Right and left eigenvectors are bi-orthogonal, i.e. despite $< L_n|L_m >\neq 0$ and $< R_n|R_m >\neq 0$, $< L_n|R_m >= \delta_{nm}$
- "Kernel" $\hat{K}_N = \sum_{i=1}^N |R_i >< L_i|$ is a projection operator due to bi-orthogonality, $\hat{K}_N^2 = \hat{K}_N$. Since we have also the closure relation ($\sum_{i=1}^\infty |R_i >< L_i| = \mathbf{1}$), we may try to repeat the "deformation" trick $\mathbf{1} \to \hat{\mathbf{K}}$ even in the cases beyond the S-L.

Example - Product of M Wishart matrices

[Akemann, Ipsen, Kieburg, 2014]

$$\mathcal{H} = z\frac{d}{dz} - \frac{d}{dz}\prod_{j=1}^{M}\left(z\frac{d}{dz} + \nu_j\right)$$

"Schroedinger eq." reads $\mathcal{H}|R_k >= k|R_k >$, and explicitly

$$< x|R_k >= G^{1,0}_{1,M+1}\left(\begin{matrix} k+1 \\ 0, -\nu_M, \ldots, -\nu_1 \end{matrix}\middle| x\right)$$

v_i measure rectangularity of Wisharts, $G^{\cdots}_{\cdots}$ - Meijer function.
From $< f|\mathcal{H}g >=< \mathcal{H}^{\dagger} f|g >$ we read out

$$\mathcal{H}^{\dagger} = -z\frac{d}{dz} - 1 + (-1)^M\frac{d}{dz}\prod_{j=1}^{M}\left(z\frac{d}{dz} - \nu_j\right).$$

with explicit solution for $< L_k|\mathcal{H}^{\dagger} = k < L_k|$

$$< L_k|x >= G^{M,1}_{1,M+1}\left(\begin{matrix} -k \\ \nu_M, \ldots, \nu_1, 0 \end{matrix}\middle| x\right)$$

Example 1 - Product of M Wishart matrices - cont

- "Halloween hat" singularity for the product of M Wisharts $\rho(r) \sim r^{-M/(M+1)}$ dictates microscopic scaling at the origin i.e. $z = Ns$.
- The Sch. equation leads therefore to the deformation (bound)

$$\mathcal{H}(z) \to \Delta_{\vec{\nu}}^{(M+1)}(s) \equiv -\frac{d}{ds}\prod_{j=1}^{M}\left(s\frac{d}{ds}+\nu_j\right) \leqslant 1.$$

- To unravel this bound we use the pair of *Narain transforms*.

$$g(s) = \int_0^\infty k(s,t)f(t)dt, \qquad f(t) = \int_0^\infty h(t,y)g(y)dy,$$

where the integral kernels read

$$\begin{aligned} k(s,t) &= 2\gamma x^{\gamma-1/2} G^{m,p}_{p+q,m+n}\left(\begin{matrix} a_1,\ldots,a_p,b_1,\ldots,b_q \\ c_1,\ldots,c_m,d_1,\ldots,d_n \end{matrix}\middle| (st)^{2\gamma}\right), \\ h(y,t) &= 2\gamma x^{\gamma-1/2} G^{n,q}_{p+q,m+n}\left(\begin{matrix} -b_1,\ldots,-b_q,-a_1,\ldots,-a_p \\ -d_1,\ldots,-d_n,-c_1,\ldots,-c_m \end{matrix}\middle| (yt)^{2\gamma}\right). \end{aligned}$$

Universal hard kernel for the product of Wisharts

- In our case, kernels read

$$k(s,y) = G^{M,0}_{0,M+1}\left(\begin{matrix} - \\ \nu_1,\ldots,\nu_M,0 \end{matrix}\middle| sy\right), \qquad h(y,t) = G^{1,0}_{0,M+1}\left(\begin{matrix} - \\ 0,-\nu_1,\ldots,-\nu_M \end{matrix}\middle| ty\right).$$

- in dual to s variable t, the bound $\Delta^{(M+1)}_{\vec{\nu}}(s) \leqslant 1$ reads simply $t \leqslant 1$
- Identity operator $g(x) = \int_0^\infty [\int_0^\infty k(x,t)h(t,y)dt]g(y)dy$ gets deformed onto

$$\mathbf{P}[g(x)] = \int_0^\infty \left[\int_0^1 k(x,t)h(t,y)dt\right] g(y)dy$$

- Hence the kernel reads explicitly

$$K^{hard}_M(x,y) = \int_0^1 G^{1,0}_{0,M+1}\left(\begin{matrix} - \\ 0,-\nu_1,\ldots,-\nu_M \end{matrix}\middle| sx\right) G^{M,0}_{0,M+1}\left(\begin{matrix} - \\ \nu_1,\ldots,\nu_M,0 \end{matrix}\middle| sy\right) ds.$$

  in agreement with [Kuijlaars, Zhang, 2014].
- For $M=1$, $G^{1,0}_{0,2}\left(\begin{matrix} - \\ \nu,0 \end{matrix}\middle| x\right) = x^{\nu/2} J_\nu(2\sqrt{x})$, so one recovers the Bessel Kernel. Narain transform generalizes Hankel transform.

Example 2 - Muttalib-Borodin Ensemble

- $P(\lambda) \sim \Delta(\lambda)\Delta(\lambda^\theta)\prod_{k=1}^{N} \lambda_k^\alpha e^{-\lambda_k}$
  where $\alpha \geqslant -1$ and $\theta \geqslant 0$.
- $\theta = 2$ - 3rd order non Hermitian diff. equation [Spencer, Fano (1951)] (paper on X-rays through matter (sic!))
- General θ: Duality between product of M Wisharts and M-B:

$$M \quad \leftrightarrow \quad \theta \tag{1}$$

$$\nu_i = T_i - N_i \quad \leftrightarrow \quad \nu_i = \frac{i}{M} - 1, \quad \text{where} \quad i = 1, \ldots, M$$

- **same kernel** as Wishart product kernel, by consequitive operations
  1. Microscopic scaling $x = uN^{-\frac{1}{\theta}}$
  2. Large N limit
  3. Change of the variables $u = \theta s^{\frac{1}{\theta}}$
     in agreement with [Kuijlaars, Stivigny, 2014]
- Analogy to Borodin-like duality similar to relation between Laguerre-generalized Hermite

Summary

- Insights from QM offer a pedagogical way to understand Borodin-Olshanski method and provide an easy alternative to advanced tools alike Plancherel-Rotach limit of orthogonal polynomials or asymptotics of Riemann-Hilbert problem
- (?) Possibility of systematic extensions of S-L problem (standard approach is based either on replacement of differential operators by difference operators (Askey-Wilson scheme) or higher order OPS (Bochner-Krall))
- (?) QM insights for the general $\beta \neq 2$?
- (?) Generalization for non-hermitian systems?

RMT faces Dataism [S. Lohr, 2015; Y. N. Harari, 2016]

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