On the persistence probability for random truncated orthogonal matrices and Kac polynomials

Mihail Poplavskyi
in collaboration with: M. Gebert (QMUL, London), G. Schehr (LPTMS, Orsay)

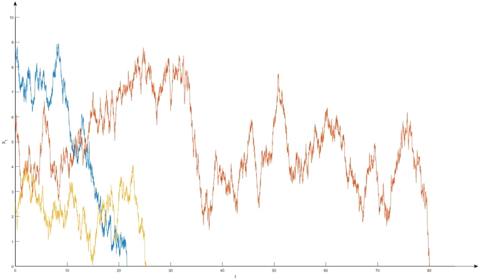
Department of Mathematics, King's College London

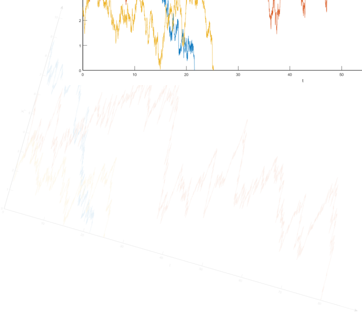
Random Matrices, Integrability and Complex Systems,
October 5, 2018

EPSRC
Engineering and Physical Sciences Research Council

erc
European Research Council
Established by the European Commission

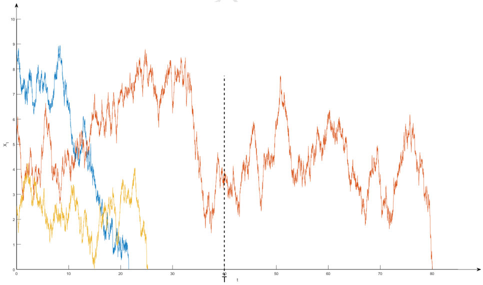
Persistence probability





$t \in [0, T]$

$K_n(z) = \sum_{k=0}^{n} a_k z^k$



Let $\{X_t\}_{t\geq 0}$ be a real valued stochastic process, then

$$p_T = \mathbb{P}\left[X_t > 0, \forall t \in [0, T]\right].$$

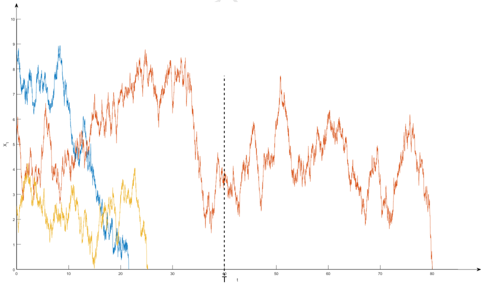
is a persistence probability of X_t up to time T.

$$t_0 = \inf\{t \geq 0 : X_t \leq 0\}$$

$$p_T = \mathbb{P}[t_0 \geq T] \cdot \mathbb{P}[X_0 > 0]$$

What is the typical behaviour of p_T?

Persistence probability



Let $\{X_t\}_{t\geq 0}$ be a real valued stochastic process, then

$$p_T = \mathbb{P}\left[X_t > 0, \forall t \in [0,T]\right].$$

is a persistence probability of X_t up to time T.

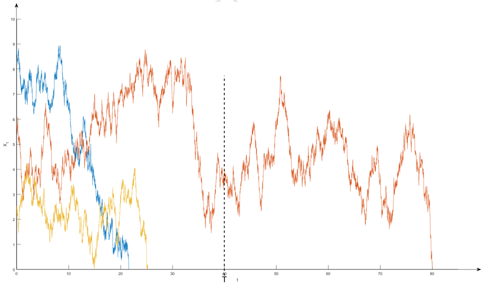
$$t_0 = \inf\{t \geq 0 : X_t \leq 0\}$$

$$p_T = \mathbb{P}\left[t_0 \geq T\right] \cdot \mathbb{P}\left[X_0 > 0\right]$$

What is the typical behaviour of p_T?

- Discrete random walk, $X_{n+1} - X_n$ are i.i.d. $\Rightarrow p_N \propto N^{-1/2}$.

Persistence probability



Let $\{X_t\}_{t\geq 0}$ be a real valued stochastic process, then

$$p_T = \mathbb{P}\left[X_t > 0,\ \forall t \in [0, T]\right].$$

is a persistence probability of X_t up to time T.

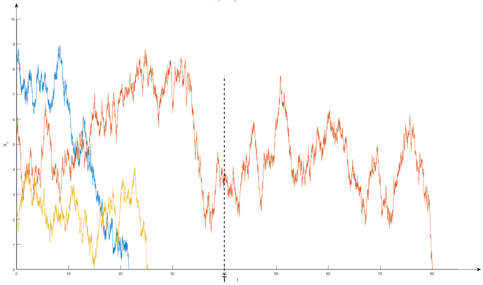
$$t_0 = \inf\{t \geq 0 : X_t \leq 0\}$$

$$p_T = \mathbb{P}\left[t_0 \geq T\right] \cdot \mathbb{P}\left[X_0 > 0\right]$$

What is the typical behaviour of p_T?

- Discrete random walk, $X_{n+1} - X_n$ are i.i.d. $\Rightarrow p_N \propto N^{-1/2}$.
- Continuous time, $\{B_t\}_{t\geq 0} \Rightarrow p_T \sim T^{-1/2}$.

Persistence probability



Let $\{X_t\}_{t\geq 0}$ be a real valued stochastic process, then

$$p_T = \mathbb{P}\left[X_t > 0,\, \forall t \in [0, T]\right].$$

is a persistence probability of X_t up to time T.

$$t_0 = \inf\{t \geq 0 : X_t \leq 0\}$$

$$p_T = \mathbb{P}[t_0 \geq T] \cdot \mathbb{P}[X_0 > 0]$$

What is the typical behaviour of p_T?

- Discrete random walk, $X_{n+1} - X_n$ are i.i.d. $\Rightarrow p_N \propto N^{-1/2}$.
- Continuous time, $\{B_t\}_{t\geq 0} \Rightarrow p_T \sim T^{-1/2}$.
- GSP with bounded spectral measures $\Rightarrow p_T \propto e^{-\theta T}$.

Guess: $p_T \propto T^{-\theta}$,or $p_T \propto e^{-\theta T}$, $\quad T \to \infty$?.

Motivation and known results

- Physics: electrons in matter are modelled by zeros of GSP; non-equilibrium systems: diffusive, spin systems; diffusion equation with random IC. (Dembo, Majumdar, Mukhterjee, Schehr)

Motivation and known results

- Physics: electrons in matter are modelled by zeros of GSP; non-equilibrium systems: diffusive, spin systems; diffusion equation with random IC. (Dembo, Majumdar, Mukhterjee, Schehr)
- Finances: reliability theory; stocks trading (price "=" Gaussian Process).

Motivation and known results

- Physics: electrons in matter are modelled by zeros of GSP; non-equilibrium systems: diffusive, spin systems; diffusion equation with random IC. (Dembo, Majumdar, Mukhterjee, Schehr)
- Finances: reliability theory; stocks trading (price "=" Gaussian Process).
- Mathematics:

Motivation and known results

- Physics: electrons in matter are modelled by zeros of GSP; non-equilibrium systems: diffusive, spin systems; diffusion equation with random IC. (Dembo, Majumdar, Mukhterjee, Schehr)
- Finances: reliability theory; stocks trading (price "=" Gaussian Process).
- Mathematics:
  1. Gaussian analytic functions and their zeros (Sodin, Tsirelson, Nishry, Hough, Krishnapur, Peres, Virag, etc.)

Motivation and known results

- Physics: electrons in matter are modelled by zeros of GSP; non-equilibrium systems: diffusive, spin systems; diffusion equation with random IC. (Dembo, Majumdar, Mukhterjee, Schehr)
- Finances: reliability theory; stocks trading (price "=" Gaussian Process).
- Mathematics:
  1. Gaussian analytic functions and their zeros (Sodin, Tsirelson, Nishry, Hough, Krishnapur, Peres, Virag, etc.)
  2. Random polynomials (Dembo, Zeitouni, Poonen, Shao) with applications to diffusion equation (Dembo, Mukhterjee, Majumdar, Schehr).

Motivation and known results

- Physics: electrons in matter are modelled by zeros of GSP; non-equilibrium systems: diffusive, spin systems; diffusion equation with random IC. (Dembo, Majumdar, Mukhterjee, Schehr)
- Finances: reliability theory; stocks trading (price "=" Gaussian Process).
- Mathematics:
  1. Gaussian analytic functions and their zeros (Sodin, Tsirelson, Nishry, Hough, Krishnapur, Peres, Virag, etc.)
  2. Random polynomials (Dembo, Zeitouni, Poonen, Shao) with applications to diffusion equation (Dembo, Mukhterjee, Majumdar, Schehr).
  3. Hyperbolic GAFs and sinc-kernel: estimates on persistence probability (Antezana, Buckley, Marzo, Olsen).

Motivation and known results

- Physics: electrons in matter are modelled by zeros of GSP; non-equilibrium systems: diffusive, spin systems; diffusion equation with random IC. (Dembo, Majumdar, Mukhterjee, Schehr)
- Finances: reliability theory; stocks trading (price "=" Gaussian Process).
- Mathematics:
  1. Gaussian analytic functions and their zeros (Sodin, Tsirelson, Nishry, Hough, Krishnapur, Peres, Virag, etc.)
  2. Random polynomials (Dembo, Zeitouni, Poonen, Shao) with applications to diffusion equation (Dembo, Mukhterjee, Majumdar, Schehr).
  3. Hyperbolic GAFs and sinc-kernel: estimates on persistence probability (Antezana, Buckley, Marzo, Olsen).
  4. GSP on the lattice in connection to nodal lines of spherical harmonics (Krishna, Krishnapur).

Still no known persistence constants for general GSP!

Motivation and known results

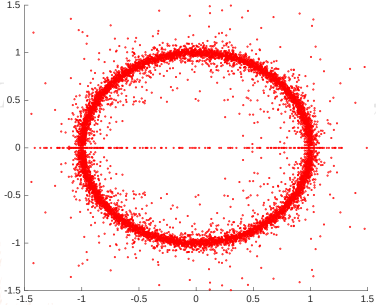
- Physics: electrons in matter are modelled by zeros of GSP; non-equilibrium systems: diffusive, spin systems; diffusion equation with random IC. (Dembo, Majumdar, Mukhterjee, Schehr)
- Finances: reliability theory; stocks trading (price "=" Gaussian Process).
- Mathematics:
  1. Gaussian analytic functions and their zeros (Sodin, Tsirelson, Nishry, Hough, Krishnapur, Peres, Virag, etc.)
  2. Random polynomials (Dembo, Zeitouni, Poonen, Shao) with applications to diffusion equation (Dembo, Mukhterjee, Majumdar, Schehr).
  3. Hyperbolic GAFs and sinc-kernel: estimates on persistence probability (Antezana, Buckley, Marzo, Olsen).
  4. GSP on the lattice in connection to nodal lines of spherical harmonics (Krishna, Krishnapur).
  5. GSP via spectral measure (Feldheim-s, Jaye, Nazarov, Nitzan).

Still no known persistence constants for general GSP!

Kac polynomials.

Kac polynomials are the polynomials with i.i.d. random coefficients, i.e.

$$K_n(z) = \sum_{k=0}^{n} a_k z^k, \quad \text{where } \{a_k\}_{k=0}^{n} \text{ i.i.d. real random variables.}$$



Find the distribution of *random* roots $\{z_k\}_{k=1}^{n}$. How many of them are real? $[\mathcal{N}_{\mathbb{R}}(n)]$ Persistence probability $p_n = \mathbb{P}[K_n(x) \neq 0, \forall x \in \mathbb{R}]$?

Some obvious symmetries: $z \to 1/z$; if a_0 has a symmetric distribution $z \to -z$.

Kac polynomials.

Kac polynomials are the polynomials with i.i.d. random coefficients, i.e.

$$K_n(z) = \sum_{k=0}^{n} a_k z^k, \quad \text{where } \{a_k\}_{k=0}^{n} \text{ i.i.d. real random variables.}$$

- Littlewood & Offord, '38, '39: $(\log\log\log n)^{-1} \ll \mathcal{N}_{\mathbb{R}}(n) \log^{-1} n \ll \log n$;
- Kac, '43: if $a_0 \sim N(0,1)$, then $\mathbb{E}[\mathcal{N}_{\mathbb{R}}(n)] = \left(\frac{2}{\pi} + o(1)\right) \log n$;
- Erdos & Offord, '56, Ibragimov & Maslova, '68, '71: $\mathbb{E}[\mathcal{N}_{\mathbb{R}}(n)]$, $\mathrm{Var}[\mathcal{N}_{\mathbb{R}}(n)]$ universality for a wide class of distributions;
- Littlewood & Offord, '39: $p_n = O\left(\log^{-1} n\right)$;
- Dembo, Poonen, Shao, Zeitouni '02: $p_{2n} \sim n^{-4\theta}$, for some explicit θ;
- Tao & Vu, '15: Local universality of roots distribution;
- Bleher & Di, '97: All correlation functions for real roots.
- Matsumoto & Shirai, '13: Pfaffian structure for random series.

Main problem: can one find the value of θ?

Kac polynomials as stochastic process.

Let $a_0 \sim N(0,1)$, then following Dembo, Poonen, Shao, Zeitouni '02 one can consider $K_n(x)$ as a Gaussian Stochastic Process (GSP) with x being a time.

$$\mathrm{Cov}\left[K_n(x), K_n(y)\right] = \frac{1-(xy)^{n+1}}{1-xy}.$$

In the limit $n \to \infty$ one introduces exponential time $x = 1 - e^{-t}$ and a proper scaling to *"obtain"* stationary GSP X_t with covariance function

$$R\left(t\right) = \mathrm{sech}\left(t/2\right).$$

Persistence probability for X_t decays exponentially with

$$\theta = -4 \lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\left[\sup_{0\le t\le T} X_t < 0\right].$$

It was shown by authors:

- $\theta \in [0.1, 0.5]$ (theoretically).

Kac polynomials as stochastic process.

Let $a_0 \sim N(0,1)$, then following Dembo, Poonen, Shao, Zeitouni '02 one can consider $K_n(x)$ as a Gaussian Stochastic Process (GSP) with x being a time.

$$\mathrm{Cov}\left[K_n(x), K_n(y)\right] = \frac{1-(xy)^{n+1}}{1-xy}.$$

In the limit $n \to \infty$ one introduces exponential time $x = 1 - e^{-t}$ and a proper scaling to *"obtain"* stationary GSP X_t with covariance function

$$R(t) = \mathrm{sech}\,(t/2)\,.$$

Persistence probability for X_t decays exponentially with

$$\theta = -4 \lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\left[\sup_{0\le t\le T} X_t < 0\right].$$

It was shown by authors:

- $\theta \in [0.1, 0.5]$ (theoretically).
- $\theta \approx 0.1875 \pm 0.01$ (numerically).

Kac polynomials as stochastic process.

Let $a_0 \sim N(0,1)$, then following Dembo, Poonen, Shao, Zeitouni '02 one can consider $K_n(x)$ as a Gaussian Stochastic Process (GSP) with x being a time.

$$\mathrm{Cov}\left[K_n(x), K_n(y)\right] = \frac{1-(xy)^{n+1}}{1-xy}.$$

In the limit $n \to \infty$ one introduces exponential time $x = 1 - e^{-t}$ and a proper scaling to *"obtain"* stationary GSP X_t with covariance function

$$R(t) = \mathrm{sech}\,(t/2)\,.$$

Persistence probability for X_t decays exponentially with

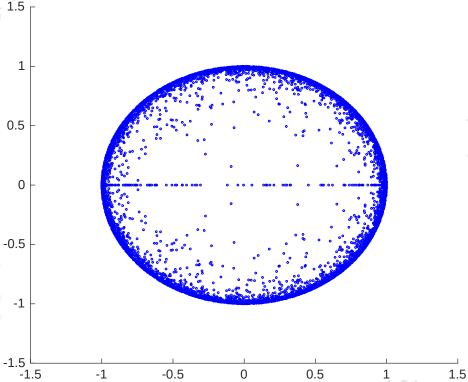
$$\theta = -4 \lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\left[\sup_{0\le t\le T} X_t < 0\right].$$

It was shown by authors:

- $\theta \in [0.1, 0.5]$ (theoretically).
- $\theta \approx 0.1875 \pm 0.01$ (numerically).
- $\dfrac{\log p_{2n}}{\log n} \to -4\theta$.

Truncated random orthogonal matrices.

We study ensemble of random matrices M_{2n} of size $2n \times 2n$ formed by top left minors of Haar distributed orthogonal matrices $O \in O(2n+1)$.



$$O = \begin{pmatrix} M_{2n} & \mathbf{u} \\ \mathbf{v}^T & a \end{pmatrix}, \qquad O\,O^T = \mathbb{I} \Rightarrow M_{2n}M_{2n}^T = \mathbb{I} - \mathbf{u}\,\mathbf{u}^T.$$

Truncated random orthogonal matrices.

We study ensemble of random matrices M_{2n} of size $2n \times 2n$ formed by top left minors of Haar distributed orthogonal matrices $O \in O(2n+1)$.

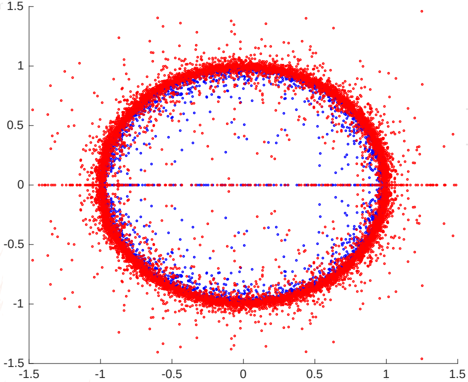


Figure: Blue dots represent eigenvalues of truncated random orthogonal matrices, while red show roots of Kac polynomials.

Truncated random orthogonal matrices.

We study ensemble of random matrices M_{2n} of size $2n \times 2n$ formed by top left minors of Haar distributed orthogonal matrices $O \in O(2n+1)$.

$$O = \begin{pmatrix} M_{2n} & \mathbf{u} \\ \mathbf{v}^T & a \end{pmatrix},$$

where $\mathbf{u}$ and $\mathbf{v}$ are column vectors of length $2n$ and a is a scalar. Let

$$X = \begin{pmatrix} I_{2n} - zM_{2n} & \mathbf{u} \\ -\mathbf{v}^T & z^{-1}a \end{pmatrix}.$$

$$\det X = \det\left(I_{2n} - zM_{2n}\right)\left(z^{-1}a + \mathbf{v}^T\left(I_{2n} - zM_{2n}\right)^{-1}\mathbf{u}\right) = z^{-1}a\det\left(I_{2n} - zM_{2n} + za^{-1}\mathbf{u}\mathbf{v}^T\right).$$

$$\det X = z^{-1}a\det\left(I_{2n} - zM_{2n}^{-T}\right) = z^{-1}a\det M_{2n}^{-1}\det\left(M_{2n} - zI_{2n}\right).$$

$$\begin{aligned} \frac{\det\left(zI_{2n} - M_{2n}\right)}{\det\left(zM_{2n} - I_{2n}\right)} &= \det M_{2n}\left(1 + za^{-1}\mathbf{v}^T\left(I_{2n} - zM_{2n}\right)^{-1}\mathbf{u}\right) \\ &= \det O\left(a + z\mathbf{v}^T\left(I_{2n} - zM_{2n}\right)^{-1}\mathbf{u}\right). \end{aligned}$$

For $|z| < 1$ one can write the r.h.s. as a series (up to a sign of $\det O$)

$$F_{2n}(z) = a + \sum_{k=1}^{\infty} z^k \mathbf{v}^T M_{2n}^{k-1}\mathbf{u} \quad \text{"} \to \text{"}\ K_{2n}(z).$$

Truncations of random orthogonal matrices.

We start with the full joint distribution of the eigenvalues of M_{2n}. Since M_{2n} is real and of even size, it has l (with l even) real eigenvalues (and possibly $l = 0$), $\lambda_1 \le \lambda_2 \le \ldots \le \lambda_l$, and $m = n - l/2$ pairs of complex conjugate eigenvalues $z_1 = x_1 + iy_1, z_2 = x_1 - iy_1, \ldots, z_{2m-1} = x_m + iy_m, z_{2m} = x_m - iy_m$ with $x_1 \le x_2 \le \ldots \le x_m$. Then ordered eigenvalues of the matrix M_{2n} conditioned to have l real eigenvalues have joint distribution (Khoruzhenko, Sommers, Zyczkowski, '10)

$$p^{(l,m)}\left(\vec{\lambda}, \vec{z}\right) = 2^m C_n \left|\Delta\left(\vec{\lambda} \cup \vec{z}\right)\right| \prod_{j=1}^{l} w(\lambda_j) \prod_{j=1}^{2m} w(z_j),$$

where C_n is a normalization constant

$$w^2(z) = (2\pi|1 - z^2|)^{-1},$$

and Δ is a Vandermonde determinant. This yields a **Pfaffian structure!** The generating function of $\mathcal{N}_n$ reads

$$\langle e^{s\mathcal{N}_n}\rangle_{M_{2n}} = \left\langle \prod_{i=1}^{2n} 1 - (1 - e^s)\chi_{\mathbb{R}}(\zeta_i) \right\rangle_{M_{2n}},$$

for $s < 0$, where the product runs over all the eigenvalues ζ_i's – both real and complex – of M_{2n}. In above, $\chi_{\mathbb{R}}(z) = 1$ if z is real and 0 otherwise and $\langle \cdots \rangle_{M_{2n}}$ denotes an average over the joint distribution.

Theorem (Sinclair, '07).
Let $\{P_i(x)\}_{i=0}^{N-1}$ be a set of monic polynomials and $\Psi : \mathbb{R}^{N\times N} \to \mathbb{R}$ is
(i) constant on similarity classes: $\Psi(AXA^{-1}) = \Psi(X)$ for all $A \in \mathbb{R}^{N\times N}$
(ii) there exists a function $\psi : \mathbb{C} \to \mathbb{R}$ such that if D is a diagonal matrix with entries $\gamma_1, \gamma_2, \dots, \gamma_N$ then $\Psi(D) = \psi(\gamma_1)\psi(\gamma_2)\cdots\psi(\gamma_N)$.
Then the average over ensemble of random matrices is given by

$$\langle \Psi \rangle = \mathcal{C}_N^{-1} \,\mathrm{Pf}\, \{\langle P_i\psi, P_j\psi\rangle_{\mathbb{R}} + \langle P_i\psi, P_j\psi\rangle_{\mathbb{C}}\}_{i,j=1}^{N}.$$

Corresponding two skew-symmetric inner products are given by

$$\langle P, Q\rangle_{\mathbb{R}} := \int_{\mathbb{R}^2} w(\alpha_1)w(\alpha_2)\, P(\alpha_1)Q(\alpha_2)\, \mathrm{sgn}(\alpha_2 - \alpha_1)\, d\alpha_1\, d\alpha_2,$$

$$\langle P, Q\rangle_{\mathbb{C}} := -2i \int_{\mathbb{C}} w(\beta)w(\overline{\beta})\, P(\overline{\beta})Q(\beta)\, \mathrm{sgn}(\mathrm{Im}(\beta))\, d\lambda_2(\beta)$$

Following Forrester, Ipsen-Forrester $P_{2j}(z) = z^{2j}, P_{2j+1}(z) = z^{2j+1} - \frac{2j}{2j+1} z^{2j-1}$.

Truncations of random orthogonal matrices.

Generating function of $\mathcal{N}_n$ can be now computed explicitly

$$\langle e^{s\mathcal{N}_n}\rangle_{M_{2n}} = \det_{0\le j,k\le n-1}\left[\delta_{j,k} - \frac{1-e^{2s}}{\pi(j+k+1/2)}\right].$$

Top left minor of infinite Hilbert matrix! $H(\lambda) = \left\{(\pi(j+k+\lambda))^{-1}\right\}_{j,k=0}^{\infty}$.

$$\det(\mathbb{I}-\alpha H_n) = \exp\left\{\mathrm{Tr}\left(\ln(\mathbb{I}-\alpha H_n)\right)\right\}, \text{with } \alpha = (1-e^{2s}).$$

Theorem (Widom, '66). The $n\times n$ matrix H_n is a top left minor of a rescaled semi-infinite Hilbert's matrix H, which has an absolutely continuous spectrum in $[0,1]$ and

$$\mu_{n,m} := \mathrm{Tr}\, H_n^m = \frac{1}{2\pi}\int_0^\infty \mathrm{sech}^m\left(\frac{\pi u}{2}\right) du \log n\,(1+o(1)),\, n\to\infty.$$

$$\langle e^{s\mathcal{N}_n}\rangle_{M_{2n}} = n^{\frac{1}{2\pi}\int_0^\infty \log\left(1-(1-e^{2s})\,\mathrm{sech}\,\frac{\pi u}{2}\right)du+o(1)}.$$

For $s<0$, the integral can be calculated explicitly as

$$\langle e^{s\mathcal{N}_n}\rangle_{M_{2n}} \sim n^{\psi(s)},\ \psi(s) = \frac{1}{8} - \frac{2}{\pi^2}\left[\cos^{-1}\left(\frac{e^s}{\sqrt{2}}\right)\right]^2.$$

Rigorous asymptotic analysis of Hilbert matrix

- Hilbert's inequality is equivalent to $\|H_n\| < 1$.
- Let $\mu_m = \lim_{n\to\infty} \mu_{n,m}$.

$$\det(\mathbb{I} - \alpha H_n) = -\sum_{k=1}^{\infty} k^{-1}\mu_{n,k} \leq -\sum_{k=1}^{M} k^{-1}\mu_{n,k} \to -\sum_{k=1}^{M} k^{-1}\mu_{k}, \forall M.$$

- $\det(\mathbb{I} - \alpha H_n) = -\sum_{k=1}^{\infty} k^{-1}\mu_{n,k} \geq \sum_{k=1}^{\infty} k^{-1}\tilde{\mu}_{n,k}$, where $\tilde{\mu}_{n,k} = \mathrm{Tr}\, \mathbb{P}_n H^k \mathbb{P}_n$.
- H is diagonalizable via Wilson polynomials, and moreover

$$\mathrm{tr}\left(\mathbb{P}_n H^m \mathbb{P}_n\right) = \int_0^{\infty} dx \left(\frac{1}{\cosh(\pi x)}\right)^m \frac{2}{\cosh(\pi x)} \sum_{n=0}^{N-1} \left|\hat{P}_n(x^2)\right|^2.$$

- Uniform asymptotics for Wilson polynomials yield $\mathrm{tr}\left(\mathbb{P}_n H^m \mathbb{P}_n\right) \to \mu_m$.

Analysis for the result.

Now taking $s \to -\infty$ one obtains the probability that M_{2n} has no real eigenvalues, using $Q_0(1, N) = \mathrm{Prob.}(\mathcal{N}_n = 0) = \lim_{s\to-\infty} \langle e^{s\mathcal{N}_n} \rangle_{M_{2n}} \sim n^{-3/8}$. From the generating function, one can also obtain the cumulants of $\mathcal{N}_n$.

$$\langle N_t^p \rangle_c \sim \kappa_p \ln t \ , \ \kappa_p = \frac{2^{p-2}}{\pi^2} \sum_{m=1}^{p} (-2)^{m-1} \Gamma^2 \left(\frac{m}{2}\right) \mathcal{S}_p^{(m)} \ ,$$

where $\mathcal{S}_p^{(m)}$ is the Stirling number of the second kind. Moreover, for large n and k, with $k/\ln n$ fixed, $p_k(n) = \mathbb{P}\left[\mathcal{N}_n = k\right]$ takes the large deviation form

$$p_k(n) \sim n^{-\varphi(k/\ln n)} \ ,$$

where the large deviation function $\varphi(x)$ is computed exactly. Its asymptotic behaviours are

$$\varphi(x) \sim \begin{cases} \frac{3}{16} + \frac{x}{2} \ln x \ , \ x \to 0 \\ \frac{1}{2\sigma^2}(x - \frac{1}{2\pi})^2 \ , \ |x - \frac{1}{2\pi}| \ll 1 \\ \frac{\pi^2}{8} x^2 \ , x \to \infty \end{cases}$$

with $\sigma^2 = 1/\pi - 2/\pi^2$.

Diffusion equation with random IC.

Another example studied by Schehr & Majumdar, Dembo & Mukherjee is the d-dimensional diffusion equation

$$\partial_t \phi(\mathbf{x}, t) = \Delta \phi(\mathbf{x}, t), \text{with } \mathbf{x} \in \mathbb{R}^d, t \in \mathbb{R}_+,$$

and initial data $\phi(\mathbf{x}, t = 0)$ given by a Gaussian random field, with zero mean and short range correlations

$$\langle \phi(\mathbf{x}, 0) \phi(\mathbf{x}', 0) \rangle = \delta^d(\mathbf{x} - \mathbf{x}').$$

The persistence $p_0(t, L)$ is the probability that $\phi(\mathbf{x}, t)$, at some fixed point $\mathbf{x}$ in space with $L = \|\mathbf{x}\|_d$, does not change sign up to time t. It takes the scaling form, for large t and large L, with t/L^2 fixed

$$p_0(t, L) \sim L^{-2\theta(d)} h(L^2/t) \ ,$$

with $h(u) \to \mathrm{const}$, when $u \to 0$ and $h(u) \propto u^{\theta(d)}$ when $u \to \infty$. Normalized process $X_t = \phi(\mathbf{0}, t)/\langle \phi(\mathbf{0}, t)^2 \rangle$ is a GSP characterised by its autocorrelation function

$$C\left(t, t'\right) = (2\sqrt{t\, t'}/(t + t'))^{d/2}, \text{when } L \to \infty.$$

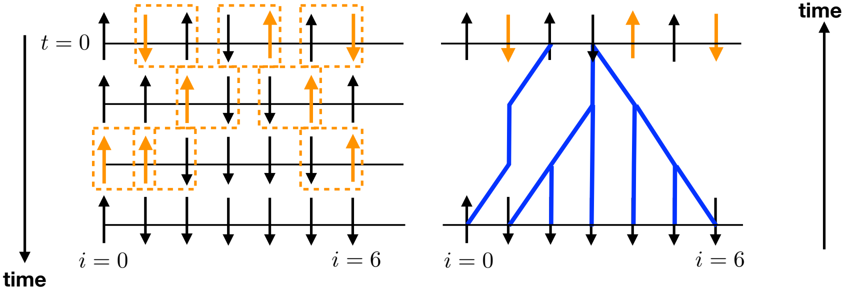
In terms of logarithmic time $T = \ln t$, $Y(T) = X(e^T)$ is a Gaussian stationary process with covariance

$$c(T) = [\mathrm{sech}(T/2)]^{d/2}.$$

Numerical simulations showed: $\theta(1) = 0.1207\ldots$, $\theta(2) = 0.1875\ldots$.

Ising spin model.

We consider the semi-infinite Ising spin chain, whose configuration at time t is given by $\{\sigma_i(t)\}_{i\geq 0}$, with $\sigma_i(t) = \pm 1$. Initially, $\sigma_i(0) = \pm 1$ with equal probability $1/2$ and, at subsequent time, the system evolves according to the Glauber dynamics at $T = 0$



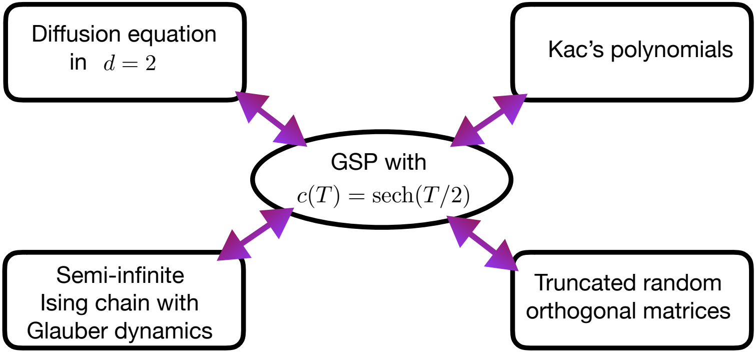
Derrida, Hakim, Pasquier '96 used mapping of q-states Potts model onto coalescing random walks to study the fraction $r(q,t)$ of spins which never flip up to time t and showed it decays like a power law

$$r\,(q,t) \sim t^{-\theta(q)},$$

where

$$\theta\,(q) = -\frac{1}{8} + \frac{2}{\pi^2}\left[\arccos\frac{2-q}{\sqrt{2q}}\right]^2, \quad \text{and } \theta\,(2) = \frac{3}{8}.$$

Stochastic models. Final remarks.



$O = \begin{pmatrix} M_{2n} & \vec{u} \\ \vec{v}^T & a \end{pmatrix}$

$T^{-1} \log \mathbb{P}\left[X_t > 0,\, t \in [0,T]\right]$

Thank you for your attention!

$K_n(z) = \sum_{k=0}^{n} a_k z^k$