Power Spectrum Analysis and Zeros of Riemann Zeta Function

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joint work with Eugene Kanzieper (HIT) and Vladimir Osipov (UCI)

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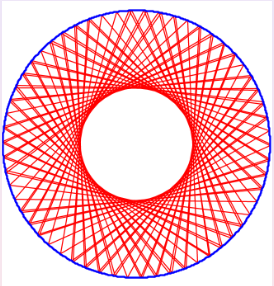
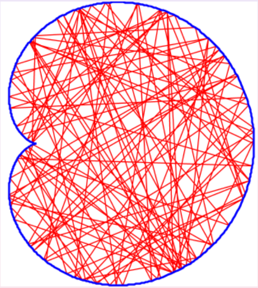
Trajectory of Classical Billiards

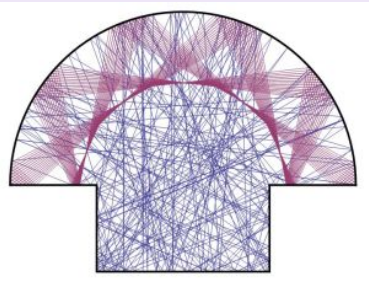
(a)



(b)



(c)

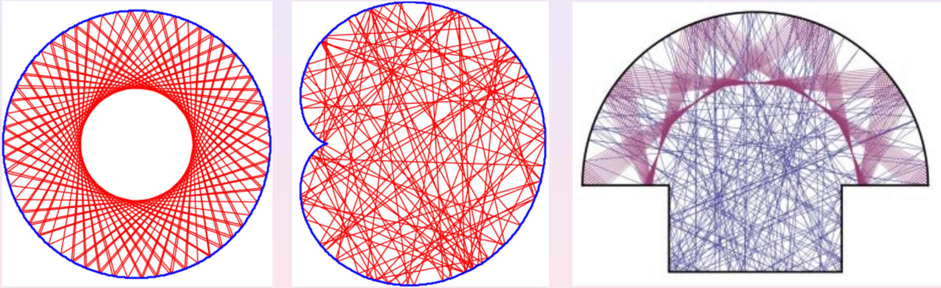


(a) & (b): © Bäcker 2007. (c): © Dettmann & Georgiou 2011.

Billiards with (a) circular shape (b) cardioid shape (c) mushroom shape boundaries. They show (a) regular geodesics, (b) chaotic geodesics and (c) a mixed phase space.

Trajectory of Classical Billiards

(a) (b) (c)

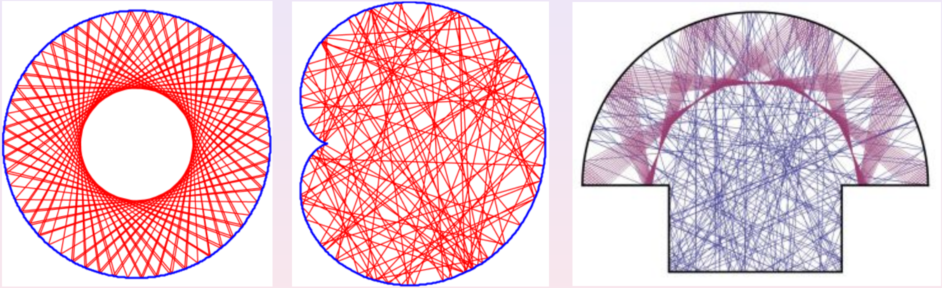


(a) & (b): © Bäcker 2007. (c): © Dettmann & Georgiou 2011.

In quantum spectra, fluctuations are known to exhibit a high degree of universality which reflects the regular or chaotic nature of the underlying classical dynamics.

Trajectory of Classical Billiards

(a) (b) (c)



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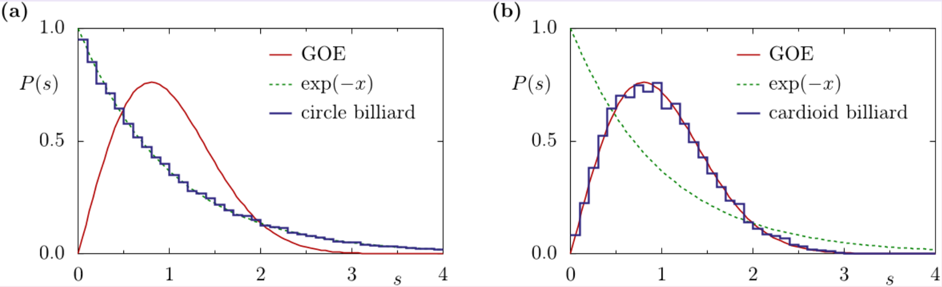
Bohigas-Giannoni-Schmit (BGS) conjecture (1984):
Statistical properties of a generic quantum system, whose classical limit is fully chaotic, coincide with those of random matrix theory.

Quantum System (Wave Billiard)

Quantum Analog: Discrete Energy Levels



Level spacing distribution for (a) circle billiard (100000 eigenvalues) and (b) cardioid billiard (11000 eigenvalues). On observes good agreement with the expected behaviour of a Poissonian random process and of the GOE, respectively.

Alternative characterization by the Power spectrum analysis of the Quantum spectra has been proposed in [1]. Long eigenlevel sequences have been interpreted as discrete-time random processes.

Power spectrum

$$S_n(\omega) = \frac{1}{n} \sum_{\ell=1}^{n} \sum_{m=1}^{n} \langle \delta\varepsilon_\ell \delta\varepsilon_m \rangle \, e^{\mathrm{i}\omega(\ell - m)}$$

- $\delta\varepsilon_\ell = \varepsilon_\ell - \langle \varepsilon_\ell \rangle$

[1] Relaño, Gómez, Molina, Retamosa, Faleiro 2002

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- $\delta\varepsilon_\ell = \varepsilon_\ell - \langle \varepsilon_\ell \rangle$
- Energy levels E_ℓ of the quantum system gets ordered and put in sequences of consecutive levels of length n.
- Unfolded eigenlevels $\varepsilon_1 \leq \cdots \leq \varepsilon_n$ fluctuate around their average positions $\langle \varepsilon_\ell \rangle = \ell\Delta$ with the mean level spacing Δ being set to unity, $\Delta = 1$.
- Average is taken over different sequences.

[1] Relaño, Gómez, Molina, Retamosa, Faleiro 2002

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Conjecture [1]

- If classical analog of a quantum system is fully integrable: Power Spectrum shows $1/\omega^2$ behavior.
- If classical analog is completely chaotic: Power Spectrum is characterized by $1/\omega$ noise.

This is expected in the large *n* limit for small frequencies, i.e. when $\omega \ll 1$.

[1] Relaño, Gómez, Molina, Retamosa, Faleiro 2002

Experiment for Regular Case: Rectangular Billiard

$$S_n(\omega = 2\pi k/n) \sim \frac{1}{k^2}$$

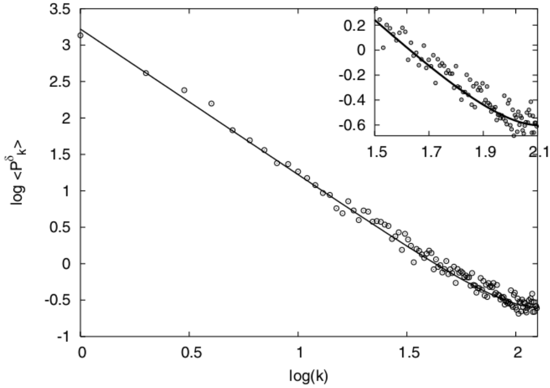

FIG. 3. Numerical average power spectrum of δ_q for a rectangular billiard, calculated using 25 sets of 256 consecutive levels, compared to the parameter free theoretical values (solid line) for integrable systems.

Random Matrix Simulation (Chaotic) vs. Poisson

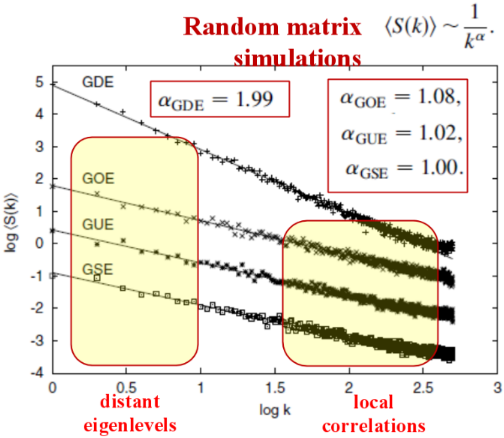

FIG. 3. Power spectrum of the δ_n function for GDE (Poisson) energy levels, compared to GOE, GUE, and GSE. The plots are displaced to avoid overlapping.

© Relaño Gómez Molina Retamosa Faleiro 2002

Simulation for Chaotic Case: Atomic Nucleus ^{34}Na

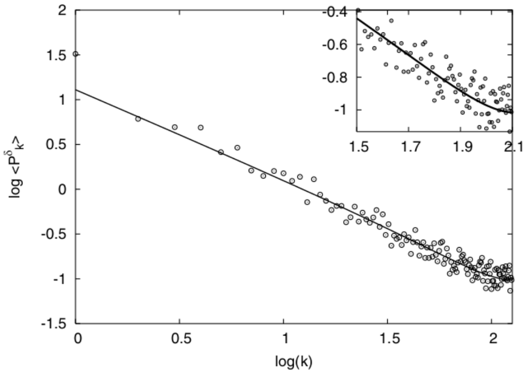

FIG. 2. Numerical average power spectrum of the δ_q function for ^{34}Na, calculated using 25 sets of 256 consecutive levels from the high level density region, compared to the parameter free theoretical values (solid line) for GOE.

- The $1/\omega$ behavior in the chaotic case and the $1/\omega^2$ behavior for the integrable case, was obtained in [2] by the form factor approximation.
- They claim that the large n asymptotic of the power spectrum can be described by the form factor $K(\omega)$ of the system by the relation

Form Factor Approximation

$$\lim_{n\to\infty} S_n(\omega) = \omega^{-2} K(\omega/(2\pi)),$$

where the spectral form factor of a quantum system is given by

$$K(\tau) = \frac{1}{n}\left(\left\langle \sum_{\ell=1}^{n}\sum_{m=1}^{n} e^{2i\pi\tau(\epsilon_\ell - \epsilon_m)}\right\rangle - \left\langle \sum_{\ell=1}^{n} e^{2i\pi\tau\epsilon_\ell}\right\rangle\left\langle \sum_{m=1}^{n} e^{-2i\pi\tau\epsilon_m}\right\rangle\right).$$

[2] Faleiro Gómez Molina Muñoz Relaño Retamosa 2004

Integrable Case

- In the regular case we have Poisson statistics for the level spacing.
- Power spectrum can be calculated directly

Power spectrum for Regular Case

$$\lim_{n\to\infty} \frac{S_n(\omega = 2\pi k/n)}{n^2} = \frac{1}{2\pi^2 k^2}, \qquad \text{for } k \in \mathbb{N}, k \ll n$$

- **Dyson's CUE.** The circular unitary ensemble on $n \times n$ matrices is defined by the joint probability density function on the eigen-angles $0 \leq \theta_k < 2\pi$

$$P_n(\theta_1, \ldots, \theta_n) = \frac{1}{n!} \left| \Delta \left(\left\{ e^{\mathrm{i}\theta_j} \right\}_{j=1}^{n} \right) \right|^2$$

where $\Delta(z_1, \ldots, z_n) = \prod_{1=j<k=n}(z_k - z_j)$ is the Vandermonde determinant.

- The p-point correlation function is given by the formula

$$R_p^{(n)}(\theta_1, \ldots, \theta_p) = \det_{1 \leq j,k \leq p} \left[\mathcal{S}_n(\theta_j - \theta_k) \right], \qquad \mathcal{S}_n(\theta) = \frac{\sin(n\theta/2)}{\sin(\theta/2)}$$

- **CUE with fixed charge at 0.** Ensemble of $(n+1) \times (n+1)$ random unitary matrices, such that one of the eigen-angles is fixed and equal to 0

$$\begin{aligned}\tilde{P}_n(\theta_1, \dots, \theta_n) &= P_{n+1}(\theta_1, \dots, \theta_n \,|\, 0) \\ &= \frac{1}{(n+1)!} \left| \Delta\left(\left\{ e^{i\theta_j} \right\}_{j=1}^{n} \right) \right|^2 \cdot \prod_{j=1}^{n} \left| 1 - e^{i\theta_j} \right|^2 .\end{aligned}$$

- Mean level spacing: $\Delta_n = \frac{2\pi}{n+1}$
- Extra ”charge“ at zero tunes eigenlevel fluctuations:

$$\langle \theta_k \rangle = k\Delta_n, \qquad k = 1, \dots, n$$

- **CUE with fixed charge at 0.** Ensemble of $(n+1)\times(n+1)$ random unitary matrices, such that one of the eigen-angles is fixed and equal to 0

$$\tilde{P}_n(\theta_1,\dots,\theta_n) = P_{n+1}(\theta_1,\dots,\theta_n \,|\, 0)$$
$$= \frac{1}{(n+1)!}\left|\Delta\left(\left\{e^{\mathrm{i}\theta_j}\right\}_{j=1}^{n}\right)\right|^2 \cdot \prod_{j=1}^{n}\left|1-e^{\mathrm{i}\theta_j}\right|^2.$$

- The mean eigenlevel density for this ensemble coincides with the two-point correlation function of CUE of the size $(n+1)$

$$\tilde{\rho}_n(\theta) = (n+1)\left(1-\frac{\sin^2((n+1)\theta/2)}{(n+1)^2\sin^2(\theta/2)}\right) = \frac{R_2^{(n+1)}(0,\theta)}{R_1^{(n+1)}(0)}, \quad \theta\in(0,2\pi).$$

Definition: Power Spectrum

$$S_n(\omega) = \frac{1}{(n+1)\Delta_n^2} \sum_{k,\ell=1}^{n} \langle \theta_k \theta_\ell \rangle_c \, z^{k-\ell}, \qquad z = e^{i\omega},$$

where the connected part is $\langle \theta_k \theta_\ell \rangle_c = \langle \theta_k \theta_\ell \rangle - \langle \theta_k \rangle \langle \theta_\ell \rangle$ and the mean level spacing is $\Delta_n = 2\pi/(n+1)$.

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- $\omega = 2\pi k/(n+1)$ with $k \in \mathbb{Z}$: power spectrum can be expressed in terms of the discrete Fourier transform $\hat{\delta}_k$ of deviations from mean of the eigenlevels $\delta_k = \theta_k - \langle \theta_k \rangle$, i.e.

$$S_n(\omega = \tfrac{2\pi k}{n+1}) = \left\langle |\hat{\delta}_k|^2 \right\rangle \qquad \text{where} \qquad \hat{\delta}_k = \frac{1}{\sqrt{n+1}} \sum_{j=1}^{n} \delta_j \, e^{-2i\pi kj/(n+1)}.$$

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From the definition: Obvious symmetries

- $S_n(\omega + 2\pi) = S_n(\omega)$
- $S_n(\omega) = S_n(-\omega)$
- Nyquist frequency is $\omega = \pi$.

Representation of Power Spectrum (for stationary spacings)

$$S_n(\omega) = \frac{2}{n}\mathrm{Re}\left(z\frac{\partial}{\partial z} - n - \frac{1-z^{-n}}{1-z}\right)\frac{z}{1-z}\int_0^\infty \mathrm{d}\phi\,\phi\,[\Phi_n(1-z,\phi) - z^n] - S_n^{(0)}(\omega),$$

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where

$$S_n^{(0)}(\omega) = \frac{1}{n}\left|\frac{1-(n+1)z^n + nz^{n+1}}{(1-z)^2}\right|^2,$$

and

$$\Phi_n(\zeta,\phi) = \sum_{\ell=0}^{n}(1-\zeta)^\ell E_n(\ell;\phi)$$
$$= \tfrac{1}{(n+1)!}\prod_{j=1}^{n}\left(\int_0^{2\pi}\frac{\mathrm{d}\theta_j}{2\pi} - \zeta\int_0^{\phi}\frac{\mathrm{d}\theta_j}{2\pi}\right)\left|1-e^{\mathrm{i}\theta_j}\right|^2\left|\Delta_n\left(\left\{e^{\mathrm{i}\theta_k}\right\}_{k=1}^{n}\right)\right|^2.$$

- $\Phi_n(\zeta,\varepsilon)$ is the generating function of $E_n(\ell;\varepsilon)$, the probability to find exactly ℓ eigenlevels below the energy ε.

- **Painlevé VI representation.** Integrals of the latter form have been studied [3] and its solution is given by

$$\Phi_n(\zeta,\phi) = \exp\left[-\int_{s=\cot(\phi/2)}^{\infty} \frac{\mathrm{d}t}{1+t^2}\left(\sigma_n(t)+t\right)\right],$$

- where $\sigma_n(t) = \sigma_n(\zeta, t)$ satisfies the Painlevé VI equation in σ-form

$$\left((1+t^2)\sigma_n''\right)^2 + 4\sigma_n'(\sigma_n - t\sigma_n')^2 + 4(\sigma_n'+1)^2(\sigma_n' + (n+1)^2) = 0$$

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- with ζ-dependent boundary condition

$$\sigma_n(t) = -t + \frac{n(n+1)(n+2)\zeta}{3\pi t^2} + \mathcal{O}(t^{-4}) \qquad \text{as} \qquad t \to \infty.$$

[3] Forrester Witte 2004

Representation as Toeplitz Determinant with Fisher-Hartwig Singularities

$$\Phi_n(\zeta,\phi) = \det_{j,k=1,\dots,n}[\mathbf{1}_n - \zeta\beta_{jk}(\phi)],$$

$$\text{where} \quad \beta_{jk}(\phi) = [\mathcal{A}^{-1/2}\,\mathcal{B}(\phi)\mathcal{A}^{-1/2}]_{jk},$$

$$\mathcal{A}_{jk} = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\pi}\,(1-\cos\theta)\mathrm{e}^{\mathrm{i}\theta(j-k)}, \qquad \mathcal{B}_{jk}(\phi) = \int_0^{\phi} \frac{\mathrm{d}\theta}{2\pi}\,(1-\cos\theta)\mathrm{e}^{\mathrm{i}\theta(j-k)}.$$

- The asymptotics of such (and more general) Toeplitz determinants $\det_{j,k=1,\dots,n}\left[\mathcal{A}_{j-k}(\phi) - \zeta\mathcal{B}_{j-k}(\phi)\right]$ with *Fisher-Hartwig* singularities has been studied in large details (see for example [4]).
- Uniform asymptotics for ϕ at the endpoints of the interval $(0, 2\pi)$ has been discussed in [5] where it is given as a solution of a Painlevé V equation.

[4] Deift Its Krasovsky 2011
[5] Claeys Krasovsky 2015

- With the help of the later, we find the following asymptotics

$$\Phi_n(\zeta,\phi) = e^{i\beta\phi}\left(\frac{\sin(\phi/2)}{\phi/2}\right)^{-2\beta^2}\exp\left(\int_0^{-in\phi}\frac{ds}{s}\sigma(s)\right) + o(1) \quad (n\to\infty)$$

  where $\beta = \omega/(2\pi) < 1/2$
- $\sigma(s)$ fulfills the σ-form of the Painlevé V equation

$$s^2(\sigma'')^2 = \left(\sigma - s\sigma' + 2(\sigma')^2\right)^2 - 4(\sigma')^2\left((\sigma')^2 - 1\right)$$

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$$\Phi_n(\zeta,\phi) = \mathrm{e}^{\mathrm{i}\beta\phi}\left(\frac{\sin(\phi/2)}{\phi/2}\right)^{-2\beta^2}\exp\left(\int_0^{-\mathrm{i}n\phi}\frac{\mathrm{d}s}{s}\sigma(s)\right) + o(1) \quad (n\to\infty)$$

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- with boundary conditions

$$\sigma(s) = -\beta s - 2\beta^2 + \frac{s\gamma(s)}{1+\gamma(s)} + \mathcal{O}(|s|^{-1+2\beta}), \qquad s\to -\mathrm{i}\infty,$$

$$\sigma(s) = \mathcal{O}(|s|\log|s|), \qquad s \to -\mathrm{i}0_+,$$

where $$\gamma(s) = \frac{1}{4}\left|\frac{s}{2}\right|^{2(-1+2\beta)}\mathrm{e}^{-\mathrm{i}|s|}\mathrm{e}^{\mathrm{i}\pi}\frac{\Gamma(2-\beta)\Gamma(1-\beta)}{\Gamma(1+\beta)\Gamma(\beta)}.$$

Result for Large Frequencies: $\omega = 2\pi\beta = \mathcal{O}(1)$

Solution is given in term of previous Painlevé V equation [6]

$$\lim_{n\to\infty} S_n(\omega) = A_\beta \,\mathrm{Im}\left[\int_0^\infty \mathrm{d}\lambda \frac{e^{i\beta\lambda}}{\lambda^{2\beta^2-1}}\left[\exp\left(\int_{-i\infty}^{-i\lambda} \mathrm{d}s \frac{\sigma(s)+\beta s+2\beta^2}{s}\right)-1\right]+B_\beta\right]$$

$$A_\beta = \frac{G_\beta}{(4\pi^2)\sin(\pi\beta)}, \qquad B_\beta = -\frac{e^{-i\pi\beta^2}}{\beta^{2-2\beta^2}}\Gamma(2-2\beta^2),$$

$$G_\beta = \prod_{j=1}^{2} G(j+\beta)G(j-\beta),$$

where $\Gamma(z)$ denotes the gamma function and $G(z)$ the Barnes G-function.

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- Solution holds if $0 < \beta < 1/2$ ($\beta = 1/2$ is the Nyquist frequency)

[6] Osipov, Kanzieper, R. 2017

small β Ansatz: Result

- We can analyze the integral for small $\beta > 0$ and we find the explicit result:

Result

$$\lim_{n\to\infty} S_n(\omega = 2\pi\beta) = \frac{1}{4\pi^2\beta} + \frac{\beta\log\beta}{2\pi^2} + \frac{\beta}{12} + \mathcal{O}(\beta^2\log\beta).$$

- One can see that it contains a correction to the $1/\omega$ law.

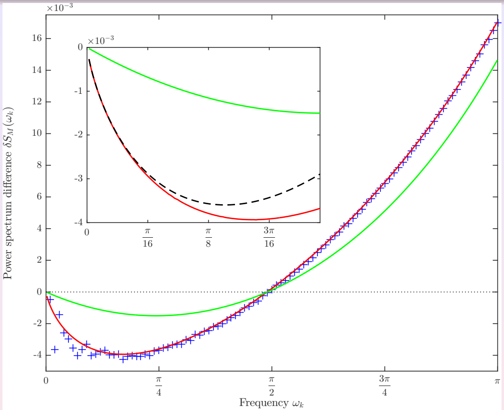


Figure: The figure shows difference of power spectrum from $(2\pi\omega)^{-1}$ plotted vs. ω. Red line represents our prediction by the the Painlevé solution. Black dashed line in the inline plot gives the small β expansion of the former. Green line is the form factor approximation. Blue crosses is the numerical simulation for unfolded eigenvalues of CUE matrcies with $n = 200$ and $M = 4 \times 10^6$ realizations.

Determinant in Fredholm Form

For small frequencies, i.e. if $\zeta = 1 - \mathrm{e}^{\mathrm{i}\omega}$ is of order $o(1)$ as n goes to infinity, we may approximate Φ_n using the following

$$\begin{aligned} \det_{j,k=1,\ldots,n}[\mathbf{1}_n - \zeta\beta_{jk}(\phi)] &= \exp\left[\mathrm{tr}_n \log(\mathbf{1}_n - \zeta\beta(\phi))\right] \\ &= \exp\left(-\sum_{\ell=1}^{\infty} \frac{\zeta^\ell}{\ell} \mathrm{tr}_n \beta^\ell(\phi)\right), \end{aligned}$$

where

$$\mathrm{tr}_n \beta^\ell(\phi) = \int_0^\phi \frac{\mathrm{d}\theta_1}{2\pi} \cdots \int_0^\phi \frac{\mathrm{d}\theta_\ell}{2\pi} \kappa_n(\theta_1, \theta_2)\, \kappa_n(\theta_2, \theta_3) \, \ldots \kappa_n(\theta_\ell, \theta_1)$$

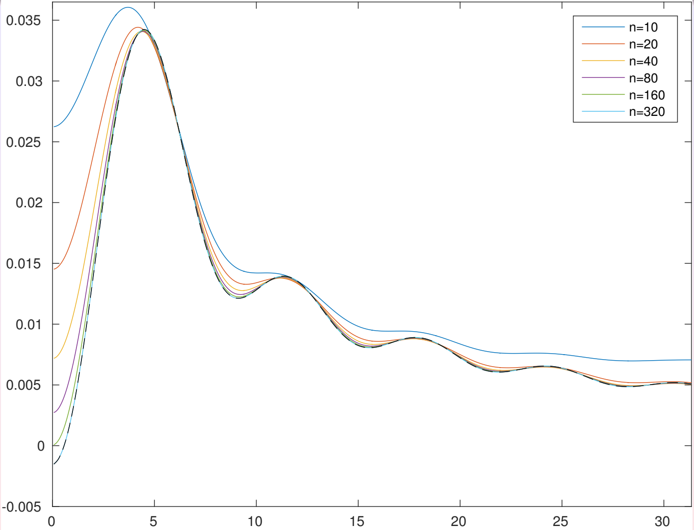
and $\kappa_n(\theta, \theta')$ is the reproducing kernel in an ensemble of $n - 1$ charges on the unit circle in presence of one fixed charges located at $\theta = 0$.

Result for Power Spectrum of Tuned CUE: Asymptotics for $\omega = \mathcal{O}(n^{-1})$

For $\omega = 2\pi k/n$ with k fixed as n goes to infinity, the power spectrum is asymptotically given by

$$S_n(\omega = c/(n+1)) = \frac{(1-\cos c)\, n \log n}{\pi^2 c^2}$$
$$+ \frac{n}{\pi^2 c^2} \left\{ (\cos c - 1) \left[\frac{\pi^2}{6} - 1 + \frac{\mathrm{d}}{\mathrm{d}c} \left(c\psi \left(\frac{c}{2\pi} \right) \right) \right] + \frac{\pi}{2}(c - \sin c) \right\} + o(n)$$

- ψ is the digamma function
- $1/\omega$ law: If $\omega = 2\pi k/(n+1)$ with $k \in \mathbb{Z}$, the power spectrum reduces in leading order to
$$\lim_{n\to\infty} S_n(\omega) = \frac{1}{2\pi\omega}.$$



Plot shows the convergence of $S_n(\omega)/n - (1 - \cos\omega)\log n/(\pi^2\omega^2)$ for the tuned CUE. The dashed line is term of order n in our asymptotic formula.

In the following analysis:

- Levels E_k are chosen by sequences of high lying consecutive zeros of the Riemann Zeta function, $\zeta(1/2 + \mathrm{i}E_k) = 0$.
- Unfolded by theoretical density prediction given by $\rho(E) = \frac{1}{2\pi}\log(E/(2\pi))$.
- For the numerical analysis we use data of up to 10 billion zeros of the Riemann Zeta function around $E \approx 10^{22}$. [7] This set of zeros has been split in M sequences of length n of consecutive zeros.

[7] We are thankful to Andrew Odlyzko who provided us with the data.

Semi-classical analysis

If one plugs the form factor $K(\omega)$ for the zeros of the Riemann Zeta function [8,9] into the form factor approximation of the power spectrum $S_\infty(\omega) = \omega^{-2} K(\omega/(2\pi))$ we get

$$S_\infty(\omega) = \frac{1}{2\pi} \sum_{m \geq 1} \sum_{p \text{ prime}} \frac{1}{m^2 p^m} \delta\left(\omega - \frac{m \log p}{\rho(E)}\right). \qquad (1)$$

- One notices that the weights of the delta distribution with $m = 1$ are much larger than the ones with $m > 1$.
- Smoothing the result in (1) and taking the limit $E \to \infty$ brings the $1/\omega$ law.

[8] Berry Keating 1999
[9] Connors Keating 2001

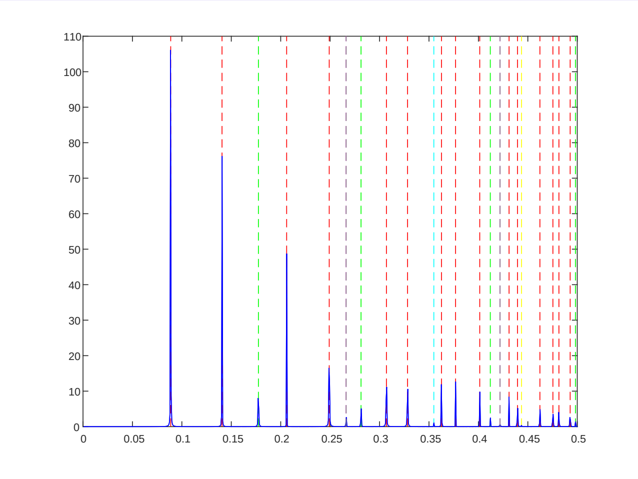


Figure: The blue line shows the numerical result for the power spectrum of the zeros of the Riemann Zeta function with $n = 10000$. The dashed lines mark the positions of the delta distributions from (1). Contributions from different m are marked with different colors.

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Smoothing by Integration

By integration of the power spectrum we define $I_n(\omega) = \int_0^\omega S_n(\tau) \mathrm{d}\tau$.

- Delta distributions in (2) become step functions.

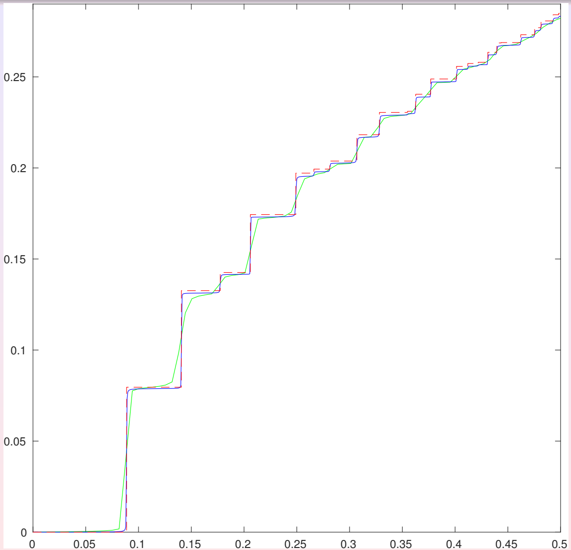

Figure: Integrated power spectrum $I_n(\omega)$ following from (2) (dashed) and for the numerical computation with $n = 1000$ (green) and $n = 10000$ (blue).

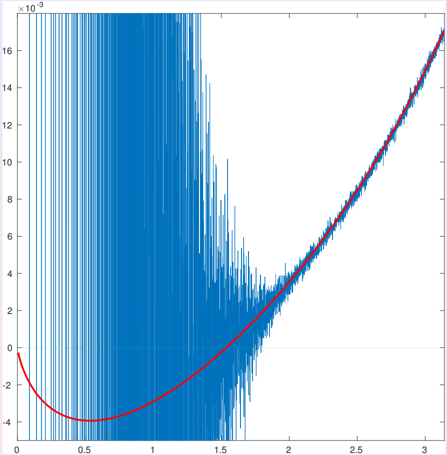

Figure: The plot shows $\delta S_n(\omega) = S_n(\omega) - (2\pi\omega)^{-1}$. The blue lines show the numerical result for the zeros of the Riemann Zeta function with $n = 10000$ averaged over $M = 100000$ samples. The red lines correspond to the analytical Painlevé solution we got for the CUE. The green zero lines indicate the $1/\omega$ law.

- The power spectrum contains statistical information about short- and long-wide correlations. While the simple form factor approximation gives the $1/\omega$ behavior straightforward, there is a deviation which is present in the CUE case as well as for the zeros of Riemann Zeta function.

Integrated Power Spectrum

For smoothing the fluctuations we define

$$\delta J_n(\omega) = \int_{\omega}^{\pi} \delta S_n(\tau) \mathrm{d}\tau$$

.

- We integrate from ω to the Nyquest frequency $\omega_{\mathrm{Ny}} = \pi$ since we are mainly interested in large frequencies and to avoid the singularity from $1/\omega$.

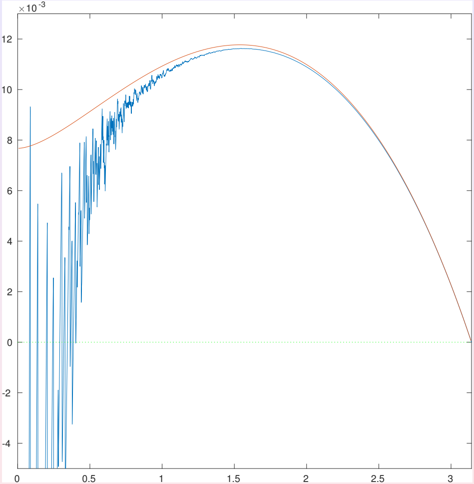

Figure: The plot shows $\delta J_n(\omega)$. The blue lines show the numerical result for the zeros of the Riemann Zeta function with $n = 10000$ averaged over $M = 100000$ samples. The red line correspond to the analytical Painlevé solution we got for the CUE. The green zero lines indicate the $1/\omega$ law.

- For finite *n*, we found an exact representation of the power spectrum in terms of a Painlevé VI transcendent.
- The large *n* asymptotics of the power spectrum for random matrices we have expressed in a parameter free form with help of a solution of a Painlevé V equation.
- For small frequencies $0 < \omega \ll 1$ it shows $1/\omega$ behavior.
- We have found a correction to the $1/\omega$ law which is also present in the behavior of the zeros of the Riemann Zeta function.

- GOE, GSE: $\beta = 1$, $\beta = 4$
- β-ensemble: Transition from $\beta = 1$ to $\beta = 2$
- Cross-over from Poisson statistic to Wigner-Dyson
- Can we find the corrections described by Forrester and Mays (2015) in the power spectrum for the Riemann Zeta function? Maybe data of more zeros or zeros around a lower E is needed?
- Conjecture for global behavior of our Painlevé V solution

Additionally, the study of the study of the power spectrum for the tuned CUE let us find the following conjecture related to the global behavior of the Painlevé V solution,

Conjecture

$$G_\beta \int_0^\infty \mathrm{d}\lambda\, \frac{\mathrm{e}^{\mathrm{i}\beta\lambda}}{\lambda^{2\beta^2}} \left[\exp\left(\int_{-\mathrm{i}\infty}^{-\mathrm{i}\lambda} \mathrm{d}s\, \frac{\sigma(s)+\beta s+2\beta^2}{s}\right)-1\right] + C_\beta = \frac{\mathrm{i}\pi}{\sin(\beta\pi)}$$

$$G_\beta = \prod_{j=1}^{2} G(j+\beta)G(j-\beta)$$

$$C_\beta = \mathrm{i}\mathrm{e}^{-\mathrm{i}\pi\beta^2}\beta^{-1+2\beta^2}\Gamma(1-2\beta^2)$$

Additionally, the study of the study of the power spectrum for the tuned CUE let us find the following conjecture related to the global behavior of the Painlevé V solution,

Conjecture

$$G_\beta \int_0^\infty \mathrm{d}\lambda \, \frac{\mathrm{e}^{\mathrm{i}\beta\lambda}}{\lambda^{2\beta^2}} \left[\exp\left(\int_{-\mathrm{i}\infty}^{-\mathrm{i}\lambda} \mathrm{d}s \, \frac{\sigma(s) + \beta s + 2\beta^2}{s} \right) - 1 \right] + C_\beta = \frac{\mathrm{i}\pi}{\sin(\beta\pi)}$$

Motivation:

$$(n+1) \int_0^{2\pi} \frac{\mathrm{d}\phi}{2\pi} \, \Phi_n(\zeta;\phi) = \frac{1 - z^{n+1}}{1 - z}$$

End of Talk

Thank you for your attention!