Product matrix processes

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1. Discrete-time determinantal processes in RMT

Example 1. Sums of independent $GUE(N)$ matrices

- $GUE(N)$:

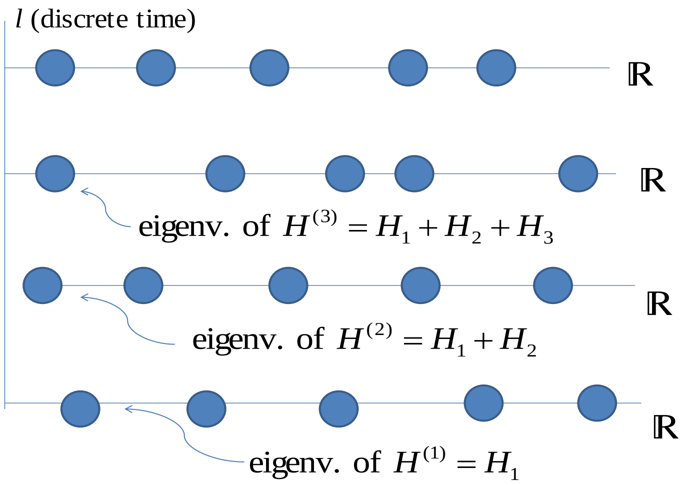
$$P(H)dH = const\ e^{-\frac{1}{2}TrH^2}\prod_{i=1}^{N} dH_{i,i}\prod_{1\le i<j\le N} dH^R_{i,j}dH^I_{i,j},$$

where $H^* = H$, $N \times N$, $H_{i,j} = H^R_{i,j} + iH^I_{i,j}$.

- Set $H^{(l)} = H_1 + \ldots + H_l$; H_1, ..., H_l-independent $GUE(N)$ matrices.

- $\left(x_1^{(l)}, \ldots, x_N^{(l)}\right)$-eigenvalues of $H^{(l)}$.

The sum matrix process



Theorem

Configurations

$$\left\{\left(l, x_j^{(l)}\right) | l \geq 1, 1 \leq j \leq N\right\}$$

form a determinantal point process on $\mathbb{N} \times \mathbb{R}$. Its correlation kernel can be written as

$$K_N(r, x; s, y) = -\frac{1}{2^{\frac{1}{2}}(s-r)^{\frac{1}{2}}\pi^{\frac{1}{2}}} e^{-\frac{(x-y)^2}{2(s-r)}} \mathbf{1}_{s>r}$$
$$+ \frac{e^{-\frac{y^2}{2s}}}{(2\pi s)^{\frac{1}{2}}} \sum_{k=0}^{N-1} \frac{1}{k!} \left(\frac{r}{4s}\right)^{\frac{k}{2}} H_k\left(\frac{x}{2^{\frac{1}{2}} r^{\frac{1}{2}}}\right) H_k\left(\frac{y}{2^{\frac{1}{2}} s^{\frac{1}{2}}}\right),$$

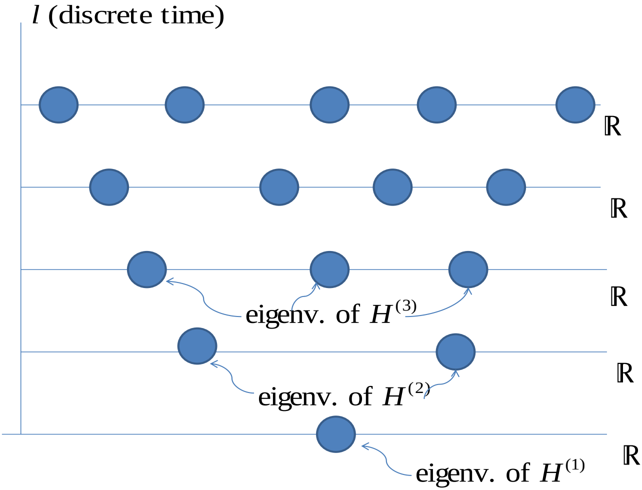
where $\{H_k(x)\}_{k=0}^{\infty}$ are the Hermite polynomials.

Example 2. The minor process (Johansson, Nordenstam)

- $H^{(1)} = H_{1,1}$; $H^{(2)} = \begin{pmatrix} H_{1,1} & H_{1,2} \\ H_{2,1} & H_{2,2} \end{pmatrix}$;

  $$H^{(3)} = \begin{pmatrix} H_{1,1} & H_{1,2} & H_{1,3} \\ H_{2,1} & H_{2,2} & H_{2,3} \\ H_{3,1} & H_{3,2} & H_{3,3} \end{pmatrix}; \ldots$$
- $H^{(l)} \in GUE(l)$;
- $\left(x_1^{(1)}, \ldots x_l^{(l)}\right)$-eigenvalues of $H^{(l)}$.

The minor process



Theorem (Johansson, Nordenstam)

- Point configurations

$$\left\{\left(l, x_j^{(l)}\right) | l \geq 1,\ 1 \leq j \leq l\right\}$$

form a determinantal point process on $\mathbb{N} \times \mathbb{R}$.

- Its correlation kernel can be written in terms of the Hermite polynomials.

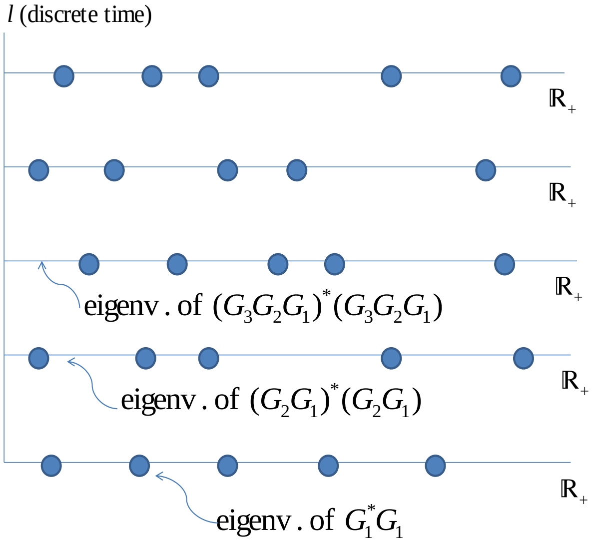
2. Product matrix processes

- $G_l = \left(g_{i,j}^{(l)}\right)_{\substack{1\leq i\leq n+\nu_l \\ 1\leq j\leq n+\nu_{l-1}}}$; $l \in \{1, 2, \ldots\}$;
  $g_{i,j}^{(l)}$-complex random variables; $\nu_0 = 0$; $\nu_l \geq 0$.

- $Y_l = (G_l \cdot \ldots \cdot G_1)^* (G_l \ldots G_1)$-$n \times n$ matrix;
  $(y_1^l, \ldots, y_n^l)$-eigenvalues of Y_l.

- Configurations $\left\{ \left(l, y_j^l\right) | l \geq 1, 1 \leq j \leq n \right\}$ form a **product matrix process**.

The product matrix process



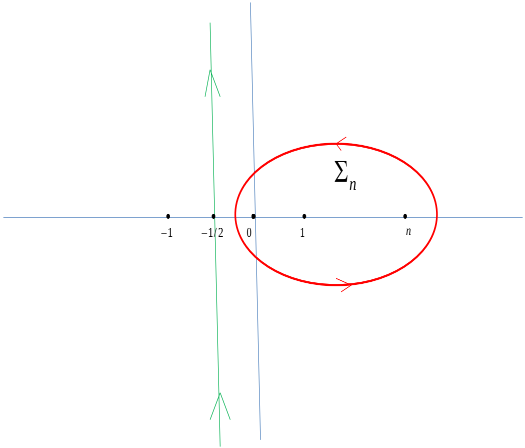
Assume that $g_{i,j}^{(l)}$ are i.i.d standard complex Gaussian variables. Then the product matrix process is determinantal. Its correlation kernel can be written as

$$K_n(r,x;s,y) = -\frac{1}{x} G^{s-r,0}_{0,s-r}\left(\begin{array}{ccc} & - & \\ \nu_{s+1}, & \ldots, & \nu_r \end{array}\middle|\frac{y}{x}\right)$$

$$+\frac{1}{(2\pi i)^2}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} du \oint_{\Sigma_n} dt \frac{\prod_{j=0}^{s}\Gamma(u+\nu_j+1)\Gamma(t-n+1)}{\prod_{j=0}^{r}\Gamma(t+\nu_j+1)\Gamma(u-n+1)}\frac{x^t y^{-u-1}}{u-t},$$

where $G^{m,0}_{0,m}\left(\begin{array}{ccc} & - & \\ \nu_1, & \ldots, & \nu_m \end{array}\middle|x\right)$ denotes the G-Meijer function.

The integration contours



Remarks

(a) If $r = s = 1$, we obtain the correlation kernel for the **Laguerre ensemble**

$$const \cdot \left(\prod_{i=1}^{n} x_i^{\nu_1} e^{-x_i} \right) \triangle^2 (x_1, \ldots, x_n)\, dx_1 \ldots dx_n,$$

where $\triangle (x_1, \ldots, x_n) = \prod_{1 \le i < j \le n} (x_j - x_i)$.

(b) If $r = s = m$, we obtain the correlation kernel for the (squared) singular values of $G_m \cdot \ldots \cdot G_1$, where G_1, ..., G_m are independent matrices with i.i.d standard complex Gaussian entries (**Akemann, Ipsen, Kieburg; Kuijlaars, Zhang**).

Product matrix processes with truncated unitary matrices

- $U_1, U_2, \ldots, U_p$-independent Haar distributed unitary matrices;
- Size $(U_k) = m_k \times m_k$, $1 \leq k \leq p$.
- Truncation of unitary matrices:

$$U_k = \begin{pmatrix} U^{(k)}_{1,1} & \ldots & U^{(k)}_{1,n+\nu_{k-1}} & U^{(k)}_{1,n+\nu_{k-1}+1} & \ldots & U^{(k)}_{1,m_k} \\ \vdots & & & & & \\ U^{(k)}_{n+\nu_k,1} & \ldots & U^{(k)}_{n+\nu_k,n+\nu_{k-1}} & U^{(k)}_{n+\nu_k,n+\nu_{k-1}+1} & \ldots & U^{(k)}_{n+\nu_k,m_k} \\ U^{(k)}_{n+\nu_k+1,1} & \ldots & U^{(k)}_{n+\nu_k+1,n+\nu_{k-1}} & U^{(k)}_{n+\nu_k+1,n+\nu_{k-1}+1} & \ldots & U^{(k)}_{n+\nu_k+1,m_k} \\ \vdots & & & & & \\ U^{(k)}_{m_k,1} & \ldots & U^{(k)}_{m_k,n+\nu_{k-1}} & U^{(k)}_{m_k,n+\nu_{k-1}+1} & \ldots & U^{(k)}_{m_k,m_k} \end{pmatrix}$$

$$\rightarrow T_k = \begin{pmatrix} U^{(k)}_{1,1} & \ldots & U^{(k)}_{1,n+\nu_{k-1}} \\ \vdots & & \\ U^{(k)}_{n+\nu_k,1} & \ldots & U^{(k)}_{n+\nu_k,n+\nu_{k-1}} \end{pmatrix}.$$

Theorem

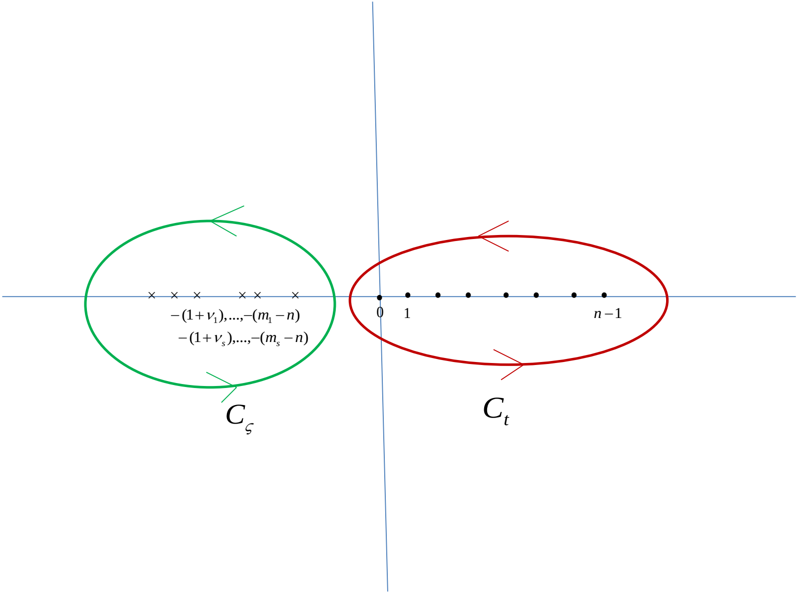
- $Y_l = (T_l \dots T_1)^* (T_l \dots T_1)$ -$n \times n$ matrix; $(y_1^l, \dots, y_n^l)$-eigenvalues of Y_l.
- Configurations $\left\{ (l, y_j^l) \middle| l \geq 1, 1 \leq j \leq n \right\}$ form a **product matrix process with truncated unitary matrices**.
- **Claim.** The product matrix process with truncated unitary matrices is determinantal. Its correlation kernel can be written as

$$K_n(r,x;s,y) = -\frac{1}{x} G^{s-r,0}_{s-r,s-r}\left(\begin{array}{ccc} m_{r+1}-n, & \dots, & m_s - n \\ \nu_{r+1}, & \dots, & \nu_s \end{array} \middle| \frac{y}{x} \right) 1_{s>r}$$

$$+ \frac{1}{(2\pi i)^2} \oint_{C_t} dt \oint_{C_\zeta} d\zeta \frac{\prod_{a=0}^{s} \Gamma(\nu_a + \zeta + 1) \prod_{a=0}^{r} \Gamma(m_a - n + t + 1)}{\prod_{a=0}^{r} \Gamma(\nu_a + t + 1) \prod_{a=0}^{s} \Gamma(m_a - n + \zeta + 1)} \frac{x^t y^{-\zeta-1}}{\zeta - t}.$$

- $r = s$ - **Kieburg, Kuijlaars and Stivigny** formula.

The integration contours



3. Product matrix processes as continuous limits of the Schur processes

Symmetric functions

- Λ-algebra of symmetric functions.
- The quotient

$$s_\lambda = s_\lambda\left(x_1, \ldots, x_n\right) = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{i,j=1}^{n}}{\det\left(x_i^{n-j}\right)_{i,j=1}^{n}}$$

  is called the **Schur function** in variables x_1, …, x_n corresponding to the Young diagram λ, and $\{s_\lambda\}$ is a basis in Λ.
- A **specialization** ϱ of Λ is an algebra homomorphism of Λ to $\mathbb{C}$. A specialization ϱ of Λ is called **nonnegative** if the Schur functions get nonnegative values.

Definition

Let p be a natural number, and let $\varrho_0^+, \ldots, \varrho_{p-1}^+, \varrho_1^-, \ldots, \varrho_p^-$ be nonnegative specializations of Λ. The probability measure

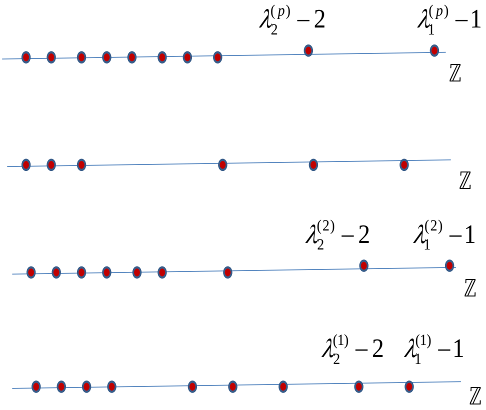
$$\begin{aligned}
&\mathrm{Prob}\left(\lambda^{(1)}, \mu^{(1)}, \ldots, \lambda^{(p-1)}, \mu^{(p-1)}, \lambda^{(p)}\right) \\
&= \frac{1}{Z_{\mathrm{Schur}}} s_{\lambda^{(1)}}\left(\varrho_0^+\right) s_{\lambda^{(1)}/\mu^{(1)}}\left(\varrho_1^-\right) s_{\lambda^{(2)}/\mu^{(1)}}\left(\varrho_1^+\right) \\
&\ldots s_{\lambda^{(p-1)}/\mu^{(p-1)}}\left(\varrho_{p-1}^-\right) s_{\lambda^{(p)}/\mu^{(p-1)}}\left(\varrho_{p-1}^+\right) s_{\lambda^{(p)}}\left(\varrho_p^-\right)
\end{aligned}$$

is called the **Schur process** (of rank p). Here Z_{Schur} is a normalization constant.

The point configurations

$$\mathcal{L}(\lambda) = \left\{ \left(1, \lambda_i^{(1)} - i\right) \right\}_{i=1}^{\infty} \cup \ldots \cup \left\{ \left(p, \lambda_i^{(p)} - i\right) \right\}_{i=1}^{\infty}$$

define a point process on $\{1, \ldots, p\} \times \mathbb{Z}$.



The Okounkov-Reshetikhin formula

Theorem

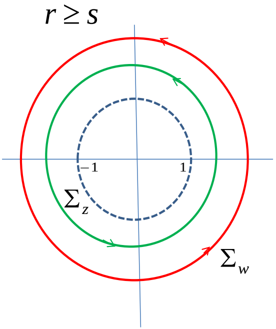
The point process is determinantal. Its correlation kernel can be written as

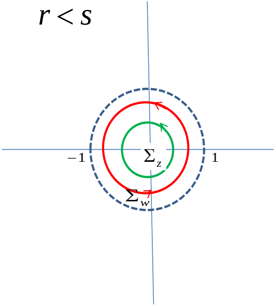
$$K_{Schur}(r,x;s,y)$$

$$= \frac{1}{(2\pi i)^2}\oint\limits_{\Sigma_z}\frac{dz}{z^{x+1}}\oint\limits_{\Sigma_w}\frac{dw}{w^{y+1}}\frac{1}{zw-1}\frac{\prod\limits_{i=r}^{p}H\left(\varrho_i^-;z\right)\prod\limits_{j=0}^{s-1}H\left(\varrho_j^+;w\right)}{\prod\limits_{j=0}^{r-1}H\left(\varrho_j^+;z^{-1}\right)\prod\limits_{i=s}^{p}H\left(\varrho_i^-;w^{-1}\right)},$$

where $H(\varrho;z) := \sum\limits_{k=0}^{\infty} h_k(\varrho)z^k,\ h_k(\varrho) := s_{(k)}(\varrho).$

The choice of integration contours depends on the time parameters r, s:





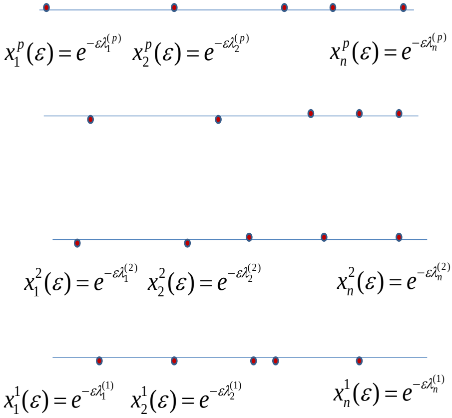
Product matrix processes as limits of the Schur process

- Assume that the specializations of the Schur process are defined by

$$\varrho_{j-1}^{+}=\left(e^{-\left(1+\nu_j\right)\epsilon},\ldots,e^{-\left(m_j-n\right)\epsilon}\right),\ 1\leq j\leq p,$$
$$\varrho_{p}^{-}=\left(1,e^{-\epsilon},\ldots,e^{-(n-1)\epsilon}\right),$$

  and all specializations $\varrho_1^-,\ldots,\varrho_{p-1}^-$ are trivial.
- Then the Schur process lives on Young diagrams with less than n rows.
- Set

$$x_j^{(l)}(\epsilon)=e^{-\epsilon\lambda_j^{(l)}};\ 1\leq j\leq n;\ 1\leq l\leq p.$$

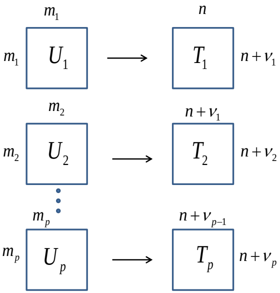

$x_1^p(\varepsilon)=e^{-\varepsilon\lambda_1^{(p)}}$ $x_2^p(\varepsilon)=e^{-\varepsilon\lambda_2^{(p)}}$ $x_n^p(\varepsilon)=e^{-\varepsilon\lambda_n^{(p)}}$
$x_1^2(\varepsilon)=e^{-\varepsilon\lambda_1^{(2)}}$ $x_2^2(\varepsilon)=e^{-\varepsilon\lambda_2^{(2)}}$ $x_n^2(\varepsilon)=e^{-\varepsilon\lambda_n^{(2)}}$
$x_1^1(\varepsilon)=e^{-\varepsilon\lambda_1^{(1)}}$ $x_2^1(\varepsilon)=e^{-\varepsilon\lambda_2^{(1)}}$ $x_n^1(\varepsilon)=e^{-\varepsilon\lambda_n^{(1)}}$

Theorem

As $\epsilon \to 0$, the point process formed by

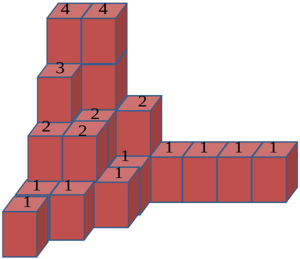
$$\left\{ \left(j, x_i^j(\epsilon) \right) \middle| j = 1, \ldots, p; i = 1, \ldots, n \right\}$$

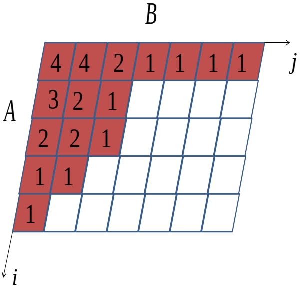
(where $x_i^j(\epsilon) = e^{-\epsilon \lambda_i^{(j)}}$) converges to the product matrix process with truncated unitary matrices T_1, …, T_p. The product matrix process is formed by the eigenvalues of $(T_l \cdot \ldots \cdot T_1)^* \cdot (T_l \cdot \ldots \cdot T_1)$, where $1 \leq l \leq p$.



4. Product matrix processes as limits of plane partitions

Plane partitions

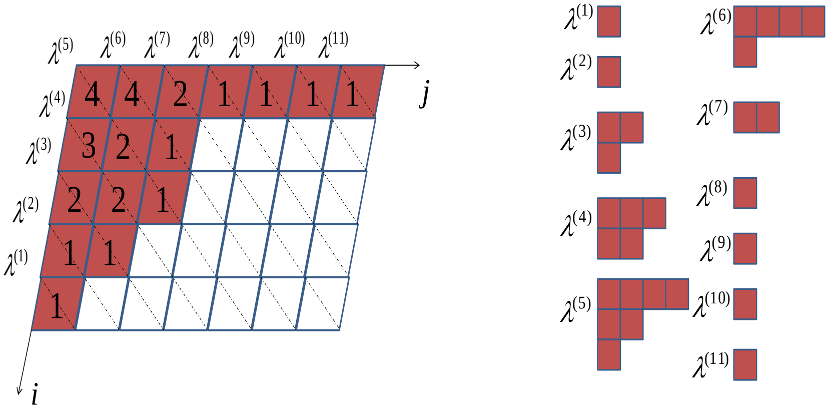




Definition

A plane partition Π with support $A \times B$ is a filling of all boxes of $A \times B$ by positive integers such that $\Pi_{i,j} \geq \Pi_{i+1,j}$ and $\Pi_{i,j} \geq \Pi_{i,j+1}$.

A plane partition as a sequence of Young diagrams



Product matrix processes as limits of plane partitions

- Assume that Π is a random plane partition with support $A \times B$,

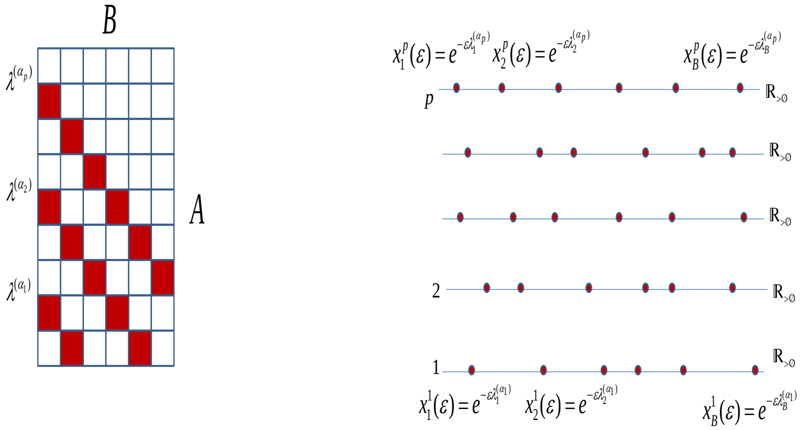
$$\text{Prob}\left\{\Pi\right\} = \frac{q^{\text{Volume}(\Pi)}}{\sum\limits_{\Pi} q^{\text{Volume}(\Pi)}}, \quad 0 < q < 1.$$

(Here the sum is over all plane partitions with support $A \times B$.)

- $\left(\lambda^{(1)}, \lambda^{(2)}, \ldots\right)$-sequence of Young diagrams associated with Π.
- $1 \leq \alpha_1 < \ldots < \alpha_p \leq A$, $q(\epsilon) = e^{-\epsilon}$, $\epsilon > 0$.
- Set

$$x_j^i(\epsilon) = e^{-\epsilon\lambda_j^{(\alpha_i)}}, 1 \leq j \leq B, 1 \leq i \leq p.$$

The particle configuration associated with a random plane partition

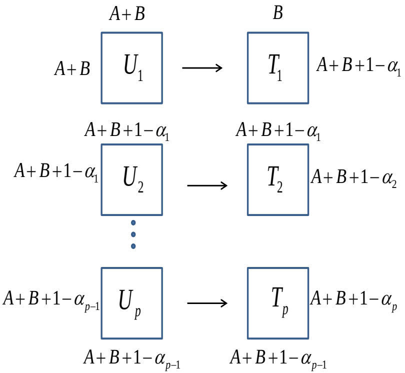


Theorem

As $\epsilon \to 0$, the point process formed by

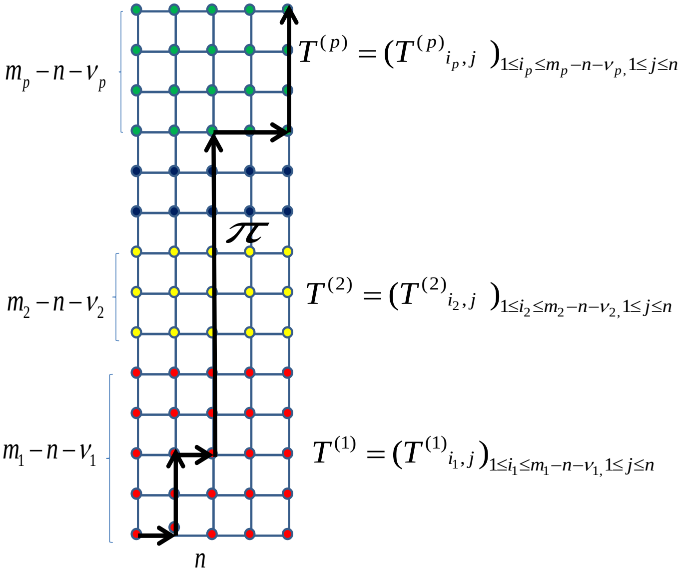
$$\left\{ \left(i, x_j^i(\epsilon) = e^{-\epsilon \lambda_j^{(\alpha_i)}} \right) \middle| i = 1, \ldots, p;\ j = 1, \ldots, B \right\}$$

converges to the product matrix process formed by eigenvalues of $(T_l \ldots T_1)^* (T_l \ldots T_1)$, where $1 \leq l \leq p$, and $T_1, \ldots, T_l$ are truncated unitary matrices.



5. Product matrix processes and last passage percolation problems

Up/right paths through arrays of random variables



The last passage time

- $T^{(1)}, T^{(2)}, \ldots, T^{(p)}$-$p$ arrays of independent random variables; size$(T^{(1)}) = (m_1 - n - \nu_1) \times n$, $\ldots$, size$(T^{(p)}) = (m_p - n - \nu_p) \times n$.
- Π-set of up/right paths from $(1,1)$ to (n,K), $K = m_1 - n - \nu_1 + \ldots + m_p - n - \nu_p$.
- **The last passage time** $\tau_{K,n}$ from $(1,1)$ to (n,K) is defined by

$$\tau_{K,n} = \max_{\pi \in \Pi} \left(\sum_{l=1}^{p} \sum_{(i_l,j) \in \pi} T^{(l)}_{i_l,j} \right)$$

- $T^{(l)}_{i_l,j}$ is the exponential random variable with the parameter $(\nu_l + i_l + j - 1)$, where $1 \leq l \leq p$, $1 \leq i_l \leq m_l - n - \nu_l$, and $1 \leq j \leq n$.

Theorem

Assume that as $n \to \infty$, $m_j - n \to \infty$, for $j = 1, \ldots, p$. Then

$$\lim_{n\to\infty}\left\{Prob\left\{\tau_{K,n} \leq \log n + \sum_{j=1}^{p} \log\left(m_j - n\right) - \log s\right\}\right\}$$

$$= \det\left(1 - \mathbb{K}\Big|_{L^2(0,s)}\right),$$

where the kernel $\mathbb{K}(x,y)$ of $\mathbb{K}$ is **the Kuijlaars-Zhang hard edge scaling limit for the product of p independent Gaussian matrices**,

$$\mathbb{K}(x,y) = \frac{1}{(2\pi i)^2}\int\limits_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} ds \int\limits_{\Sigma} dt \prod_{j=0}^{p} \frac{\Gamma\left(s+\nu_j+1\right)}{\Gamma\left(t+\nu_j+1\right)} \frac{\sin \pi s}{\sin \pi t} \frac{x^t y^{-s-1}}{s-t}.$$