Eigenvector non-orthogonality in non-Hermitian random matrices

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joint work with M. A. Nowak and
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Plan

- Why eigenvectors?
- Proper objects to look at
- Main results
- Some comments on the microscopic universality
- A brief overview on the derivation
- Conclusions and perspectives

Setting the stage: reminder from algebra

A matrix X is non-normal iff $XX^\dagger \neq X^\dagger X$.
If a non-normal matrix can be diagonalized, it possesses two sets of eigenvectors: right $|R_k\rangle$ (column) and left $\langle L_k|$ (rows), satisfying eigenequations

$$\langle L_k| X = \langle L_k| \lambda_k, \qquad X |R_k\rangle = \lambda_k |R_k\rangle$$

The diagonalization is via similarity transformation $X = S\Lambda S^{-1}$ with S and S^{-1} encoding eigenvectors $X = \sum_k |R_k\rangle \lambda_k \langle L_k|$.
The eigenvectors are not orthogonal $\langle R_k|R_l\rangle \neq \delta_{kj}$ but biorthogonal $\langle L_k|R_j\rangle = \delta_{kj}$ ($\Leftrightarrow S^{-1}S = \mathbf{1}$).
Resolution of identity $\sum_k |R_k\rangle \langle L_k| = \mathbf{1}$ ($\Leftrightarrow SS^{-1} = \mathbf{1}$).

Motivation Jacek Grela's talk

Some complicated dynamics $\dot{y}_i = f_i(y_1, \dots, y_N)$
Linearization around some fixed point y^*:

$$\frac{d}{dt}\delta y_i = \sum_{j=1}^{N} \frac{\partial f_i}{\partial y_k}\delta y_k$$

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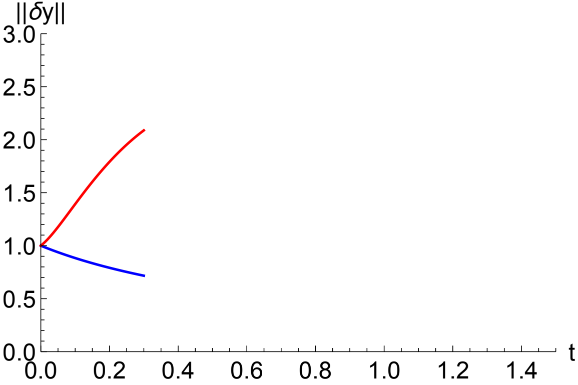

$$\delta y(0) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)^T$$

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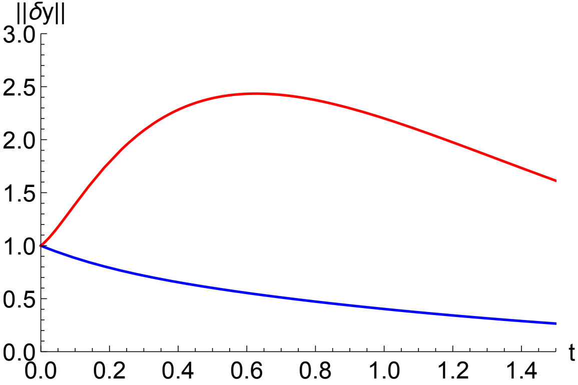

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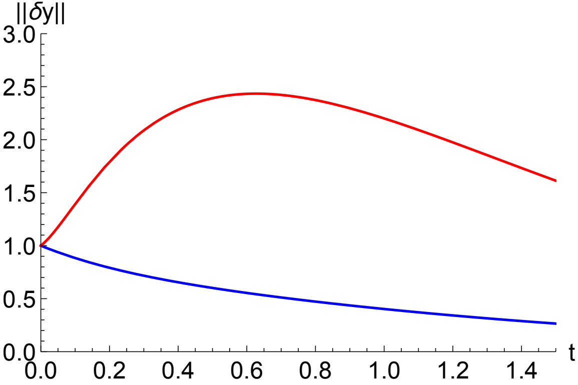


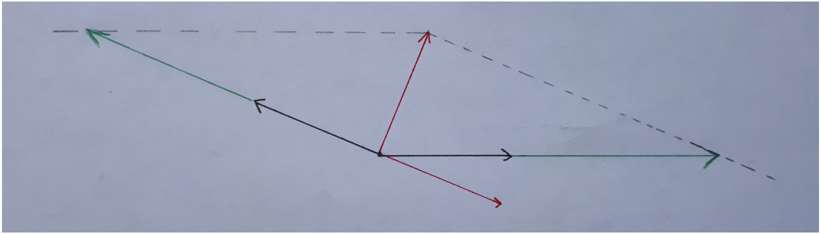
$$\delta y(0) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)^T$$

$$\begin{pmatrix} -1 & 10 \\ 0 & -2 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix}$$

Why does it happen?

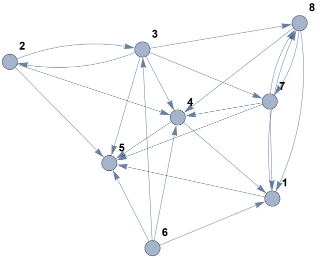
$$||\delta y(t)||^2 = \sum_{j,k=1}^{N} e^{(\lambda_j+\bar{\lambda}_k)t} \underbrace{\langle \delta y_0 | L_k \rangle} \langle R_k | R_j \rangle \underbrace{\langle L_j | \delta y_0 \rangle}.$$



For normal matrices (eigenvectors are orthogonal)

$$||\delta y(t)||^2 = \sum_{k=1}^{N} e^{2t\mathrm{Re}\lambda_k} | \langle L_k | \delta y_0 \rangle |^2.$$

Temporal changes of networks seen as perturbations



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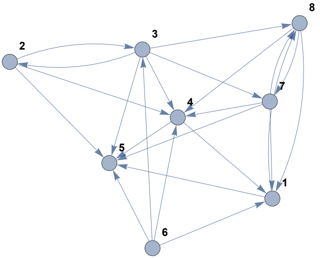
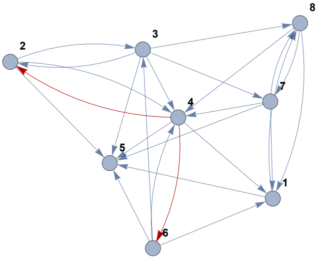
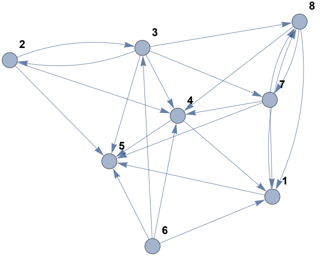
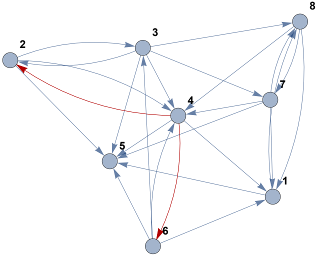




Temporal changes of networks seen as perturbations





Adjacency matrix: $A \to A' = A + P$ How does the spectrum change? In first order perturbation theory

$$\lambda_k' = \lambda_k + \langle L_k | P | R_k \rangle + \mathcal{O}(||P||^2)$$

Upper bound $|\delta \lambda_k| \leqslant ||L_k|| \cdot ||R_k|| \cdot ||P|| = ||P|| \sqrt{\langle L_k | L_k \rangle \langle R_k | R_k \rangle}$.
Eigenvalue condition number [Wilkinson 1965]

How to address the problem of eigenvectors correctly?

Biorthogonality $\langle L_k | R_j \rangle = \delta_{kj}$, completeness $\sum_k |R_k\rangle \langle L_k| = \mathbf{1}$
Invariant under rescaling $|R_k\rangle \to c_k |R_k\rangle$ and $\langle L_k| \to \langle L_k| c_k^{-1}$
The simplest invariant quantity: matrix of overlaps
$O_{ij} = \langle L_i | L_j \rangle \langle R_j | R_i \rangle$ Chalker Mehlig [1998]
Weighted density

$$D(z,w) = \left\langle \frac{1}{N} \sum_{j,k=1}^{N} O_{jk} \delta(z-\lambda_j) \delta(w-\lambda_k) \right\rangle = \tilde{O}_1(z)\delta(z-w) + O_2(z,w)$$

with

$$\tilde{O}_1(z,w) = \left\langle \frac{1}{N} \sum_k O_{kk} \delta^{(2)}(z-\lambda_k) \right\rangle \quad \left(O_1 = \frac{1}{N} \tilde{O}_1 \right),$$

$$O_2(z,w) = \left\langle \frac{1}{N} \sum_{j \neq k} O_{jk} \delta^{(2)}(z-\lambda_j) \delta^{(2)}(w-\lambda_k) \right\rangle$$

Sum rules: $\sum_j O_{ij} = 1 \Rightarrow \int d^2 w D(z,w) = \rho(z)$

Overview on the progress

- Ginibre finite N + large N Chalker, Mehlig ['98,'00]
- Quantum scattering ensemble Mehlig, Fyodorov, Frahm, Schomerus, Beenakker (et al.) ['00-'03]
- O_1 in large N for unitarily invariant matrices Nowak et al. ['99]
- Eigenvector non-orthogonality can be experimentally probed Fyodorov, Savin ['11], Legrand et al. ['14]
- Crucial role in diffusion processes on matrices Kraków group ['14], Dubach, Burgade ['18], Grela, Warchoł ['18]
- O_1 for biunitarily invariant ensembles (Single ring theorem part 2) Belinschi, Nowak, Speicher, WT ['17]
- Full distribution of O_{ii} Bourgade, Dubach ['18], Fyodorov ['17]
- O_2 in large N for unitarily invariant matrices; special case biuintarily invariant ensembles Nowak, WT ['18] $\leftarrow$ **TODAY**
- Extention to multi-point functions and calculation for the Ginibre Crawford, Rosenthal ['18]
- Determinantal structure Akemann, Zaboronski ['18?]

Main results

Complex matrices with unitary invariance $P(X) = P(UXU^\dagger)$

$$\underbrace{K(Q,P)}_{two-point} = \underbrace{G(Q)\otimes G^T(P)}_{1-point}\left(\mathbf{1}_4 + \underbrace{\Gamma(Q,P)}_{cumulants} K(\underbrace{Q,P}_{quaternions})\right)$$

Traced product of resolvents

$\mathfrak{h}(z_1,\bar{z}_2) = \left\langle \frac{1}{N}\mathrm{Tr}(z_1 - X)^{-1}(\bar{z}_2 - X^\dagger)^{-1}\right\rangle$

$$\mathfrak{h}(z_1,\bar{z}_2) = \frac{\mathfrak{g}(z_1)\bar{\mathfrak{g}}(\bar{z}_2)}{1-\mathfrak{g}(z_1)\bar{\mathfrak{g}}(\bar{z}_2)\mathcal{R}_{1\bar{1}}(\mathrm{diag}(\mathfrak{g}(z_1),\bar{\mathfrak{g}}(\bar{z}_2))}$$

Dunford-Taylor integral

$$\left\langle \frac{1}{N}\mathrm{Tr} f(X) g(X^\dagger)\right\rangle = \frac{1}{(2\pi i)^2}\oint\oint f(z_1) g(\bar{z}_2)\mathfrak{h}(z_1,\bar{z}_2) dz_1 d\bar{z}_2$$

For transients [Grela's talk] take $f = g = \exp(Xt)$.

Biunitarily invariant ensembles

pdf invariant under transformation $X \to UXV^\dagger$ with $U, V \in U(N)$. Symmetry transformations bring to the SVD canonical form $\to$ all observables are determined by the distribution of singular values. Spectrum is rotationally invariant $\rho(z,\bar{z}) = \rho(r = |z|)$.
$F(r) = 2\pi \int_0^r \rho(r) r dr, \quad r^2 = |z|^2$

- [Feinberg, Zee '97] Large N: single ring theorem
- $S_{XX^\dagger}(F(r)-1) = \frac{1}{r^2}$ Haagerup, Larsen ['00]
- Exact finite N mapping between jpdfs Kieburg, Kosters ['17]
- $O_1(r) = \frac{1}{\pi}\frac{F(r)(1-F(r))}{r^2}$ Belinschi, Speicher, Nowak, WT ['17]
- $$O_2(z_1,z_2) = \frac{1}{\pi}\partial_{\bar{z}_1}\partial_{z_2}\frac{\bar{z}_1(z_1-z_2)O_1(r_1)+z_2(\bar{z}_1-\bar{z}_2)O_1(r_2)}{|z_1-z_2|^2[F(r_1)-F(r_2)]}$$
- $\mathfrak{h}(z_1,\bar{z}_2) = \frac{1}{z_1\bar{z}_2 - r_{out}^2}$

Examples

- Ginibre

$$O_2(z_1, z_2) = \frac{-1}{\pi^2} \frac{1 - z_1 \bar{z}_2}{|z_1 - z_2|^4}$$

- Truncated unitary. $U(N+L) \to N \times N$, $N, L \to \infty$ with $\kappa = \frac{L}{N}$ fixed.

$$O_2(z_1, z_2) = \frac{1}{\pi^2} \frac{-1 + z_1 \bar{z}_2 (1 + \kappa)}{|z_1 - z_2|^4}.$$

- Spherical ensemble (ratio of two Ginibres)

$$O_2(z_1, z_2) = \frac{1}{\pi^2} \frac{-1}{|z_1 - z_2|^4}$$

- Product of 2 Ginibres

$$O_2(z_1, z_2) = -\frac{1}{\pi^2} \frac{2(|z_1| + |z_2|)(z_1 \bar{z}_2 + |z_1 z_2|) - |z_1 + z_2|^2 - 4|z_1 z_2|}{4|z_1 z_2||z_1 - z_2|^4}$$

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Towards microscopic universality

Microscopic scaling is not accessible within this framework.
Hermitian models: singularities of two-point Green's functions are heralds of non-trivial scaling limits.
Sum rule:

$$NO_1(z) + \int O_2(z,w) d^2w = \rho(z)$$

Microscopic scaling: $w = z + \frac{u}{\sqrt{N\rho(z)}}$

$$d^2w \to \frac{d^2u}{N\rho(z)}, \qquad O_2 = \frac{-1}{\pi^2}\frac{P(z,w)}{|z-w|^4} \to N^2\rho^2(z)\frac{P(z,z)}{|u|^4}$$

Explicit calculations:

$$\lim_{w\to z} P(z,w) = \frac{O_1(z)}{\rho(z)}.$$


BEWARE!
CONJECTURES
AHEAD

Conjecture

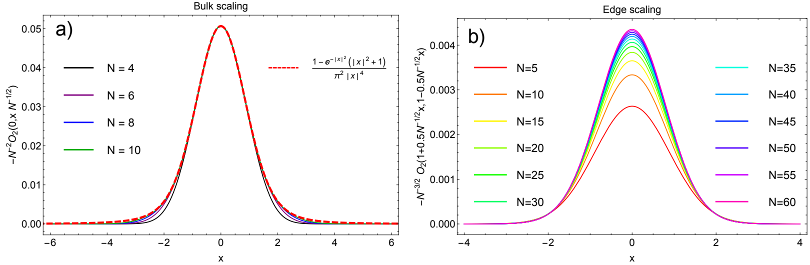
In generic complex non-Hermitian matrix models for all points in the bulk at which the spectral density does not develop singularity

$$\lim_{N\to\infty} N^{-2} O_2\left(z+\frac{x}{\sqrt{N\rho(z)}}, z+\frac{y}{\sqrt{N\rho(z)}}\right) = O_1(z)\Phi(|x-y|),$$

where

$$\Phi(|\omega|) = -\frac{1}{\pi^2\omega^4}\left(1-\left(1+|\omega|^2\right)e^{-|\omega|^2}\right)$$

Ginibre z=0 Chalker, Mehlig ['00], bulk Bourgade, Dubach ['18]



1-point functions [Janik et al., Feinberg and Zee '97]

For the spectral density $\left\langle \frac{1}{N} \sum \delta^{(2)}(z-\lambda_i) \right\rangle$ we need 2D Dirac delta. Identity $\pi\delta^{(2)}(z) = \partial_{\bar{z}} \frac{1}{z}$. Natural candidate $\mathfrak{g}(z) = \left\langle \frac{1}{N} \mathrm{Tr}(z-X)^{-1} \right\rangle$. Moment expansion valid only outside the spectrum $\to$ does not provide the distribution of eigenvalues.
Idea: regularize

$$\mathfrak{g}(z) \to g(z,w) = \left\langle \frac{1}{N} \mathrm{Tr} \frac{\bar{z} - X^\dagger}{(z-X)(\bar{z}-X^\dagger) + |w|^2} \right\rangle$$

Problem: how to deal with quadratic denominator? Linearize it

$$G(z) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \left\langle \frac{1}{N} \mathrm{bTr} \begin{pmatrix} z-X & i\bar{w} \\ iw & \bar{z}-X^\dagger \end{pmatrix}^{-1} \right\rangle \text{ [Janik et al]}$$

$$\left\langle \frac{1}{N} \mathrm{bTr} \begin{pmatrix} \epsilon & z-X \\ \bar{z}-X^\dagger & \epsilon \end{pmatrix}^{-1} \right\rangle \text{ [Feinberg, Zee]}$$

This construction can be written in the resolvent form

$$\mathcal{G} = \left\langle (Q - \mathcal{X})^{-1} \right\rangle, \qquad G(z) = \frac{1}{N}\mathrm{bTr}\mathcal{G}$$

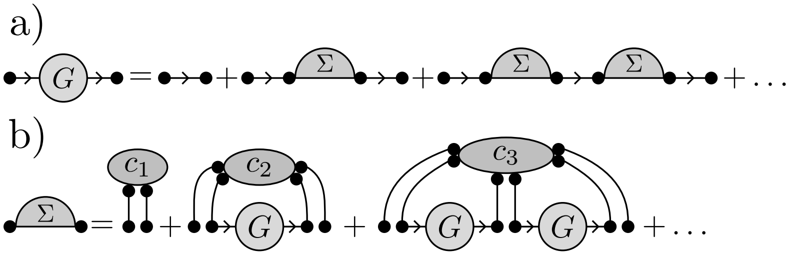
with

$$Q = \begin{pmatrix} z & i\bar{w} \\ iw & \bar{z} \end{pmatrix}, \qquad \mathcal{X} = \begin{pmatrix} X & 0 \\ 0 & X^\dagger \end{pmatrix}$$

Moment expansion

$$\mathcal{G} = Q^{-1} + \left\langle Q^{-1}\mathcal{X}Q^{-1} \right\rangle + \left\langle Q^{-1}\mathcal{X}Q^{-1}\mathcal{X}Q^{-1} \right\rangle + \dots$$

Large N limit: planar diagrams $\rightarrow$ Schwinger-Dyson equation



2-point functions

Natural candidate

$$\mathfrak{h}(z_1, \bar{z}_2) = \frac{1}{N}\mathrm{Tr}(z_1 - X)^{-1}(\bar{z}_2 - X^{\dagger})^{-1} = \frac{1}{N}\sum_{k,l} O_{kl}\frac{1}{(z_1-\lambda_k)(\bar{z}_2-\bar{\lambda}_l)}$$

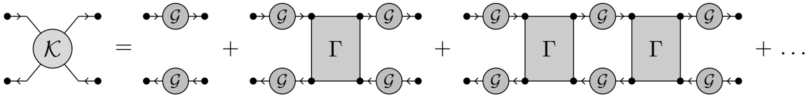
Same problems $\Rightarrow$ regularization + linearization

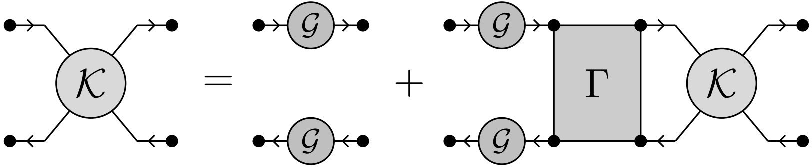
$$\mathcal{K} = \left\langle (Q-\mathcal{X})^{-1} \otimes (P^T - \mathcal{X}^T)^{-1} \right\rangle$$

+ proper contraction of indices (like a block-trace) $\Rightarrow 4 \times 4$ matrix. One of its entries is the object of our interest.

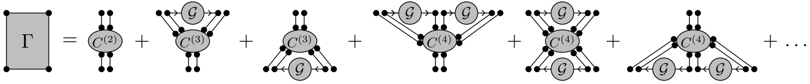
Details in Arxiv: [1801.02526]

Moment expansion → planar diagrams





$$K(Q,P) = G(Q) \otimes G^T(P)\left(\mathbf{1}_4 + \Gamma(Q,P)K(Q,P)\right)$$



Conclusions

- Non-orthogonality of eigenvectors might be of interest
- Full formalism for calculations of the two-point function in large N limit for complex non-Hermitian matrices with unitarily invariant pdf.
- Applications: work in progress. First trial [1805.03592]

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Thank you for attention