Massive Modes for Chaotic Quantum Graphs

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Random Matrices, Integrability, and Complex Systems
Yad Hashmona, October 2018

Contents:

1. Purpose

Bohigas, Giannoni and Schmit conjecture: Spectral fluctuation properties of Hamiltonian systems that are chaotic in the classical limit coincide with those of random-matrix ensemble in same symmetry class. Central element in understanding quantum chaos.

O. Bohigas, M. J. Giannoni, C, Schmit, Phys. Rev. Lett. 52 (1984) 1.

Substantial numerical evidence. Two analytical approaches to proof: Partial summation of Gutzwiller's semiclassical expansion of level density for general systems, and study of chaotic quantum graphs.

M. Sieber and K. Richter, Phys. Scr. T90 (2001) 128.

S. Müller, S. Heusler, A. Altland, P. Braun, F. Haake, New Journal of Physics 11 (2009) 103025

A. V. Andreev, O. Agam, B. D. Simons, B. L. Altshuler, Phys. Rev. Lett. 76 (1996) 3947 and Nucl. Phys. B 482 (1996) 536.

M. R. Zirnbauer, J. Math. Phys. 38 (1997) 2007, arXiv:cond.mat/9701024.

S. Gnutzmann, A. Altland, Phys. Rev. Lett. 93 (2004) 014104, Phys. Rev. E 72 (2005) 056215.

Z. Pluhar, H. A. Weidenmüller, Phys. Rev. Lett. 110 (2013) 034101, Phys. Rev. E 88 (2013) 022902, Phys. Rev. Lett. 112 (2014) 144102, J. Math. Phys.: Math. Theor. 48 (2015) 275102.

In chaotic quantum graphs supersymmetry leads to variables defined in coset space. The problem is known but all treatments so far separate universal mode and massive modes as though they were defined in ordinary vector space.

A. Altland, S. Gnutzmann, F. Haake, T. Micklitz, Rep. Prog. Phys. 78 (2015) 086001.

Present treatment takes full account of coset space structure for the case of the two-point function.

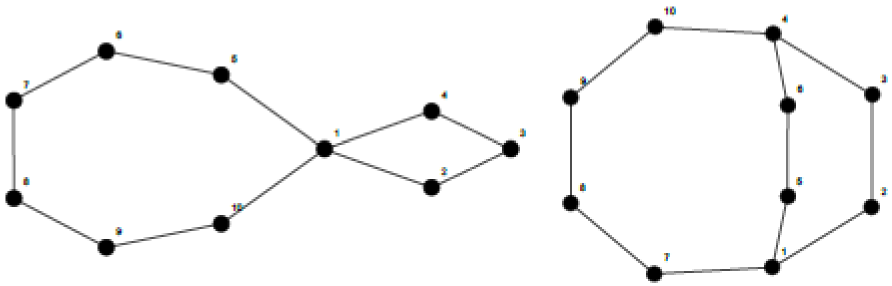
2. Chaotic Quantum Graphs. Supersymmetry

T. Kottos and U. Smilansky, Ann. Phys. 274 (1999) 76.

Quantum Graph: V vertices connected by B bonds. Connected and simple. Directed bonds with direction d labeled (b d). Schrödinger wave carries same wave number k on all bonds and a phase ϕ_{bd} that breaks T-invariance. Hermitean boundary conditions on all vertices. Incoming and outgoing waves on bonds connected to same vertex α related by unitary matrix σ^{α}. Totality of all these defines unitary bond scattering matrix $\Sigma^{(B)}_{bd,b'd'}$. Amplitude propagation on graph defined by matrix

$$\mathcal{B}_{bd,b'd'} = (\sigma_1^d \Sigma^{(B)})_{bd,b'd'}$$

where σ_1^d is first Pauli spin matrix in directional space. All bond lengths incommensurate.



Perron-Frobenius operator is

$$\mathcal{F}_{bd,b'd'} = |\mathcal{B}_{bd,b'd'}|^2$$

That operator governs relaxation of classical system towards equilibrium (equal occupation probability for all bonds). Matrix $\mathcal{F}$ is bistochastic, one eigenvalue is +1. All remaining eigenvalues obey $|\lambda_i| \leq 1$. We assume that $|\lambda_i| \leq 1 - a < 1$, even in the limit $B \to \infty$: Spectrum has a finite gap of size a. Classical relaxation is exponentially fast. Proof of universality for weaker condition as used by some authors seemingly not attainable via perturbative methods. To keep gap from closing as B increases, connectivity of graph must increase.

Unitary symmetry realized by averaging separately and independently over phases ϕ_{bd}. Consider only two-point function: Product of retarded and advanced Green functions. Using supersymmetry this is written as derivative of generating function. Average calculated using color-flavor transformation (exact). Yields supermatrices $Z_{bd;ss'}$ and $\tilde{Z}_{bd;ss'}$, both of dimension 2 where s = (B, F). Related by symmetry. M. R. Zirnbauer, J. Math. Phys. 38 (1997) 2007

Averaged two-point function is integral over all $Z_{bd;ss'}$ and $\tilde{Z}_{bd;ss'}$. Integrand carries in exponent minus the effective action (supertrace implies summation over (b d))

$$\mathcal{A} = -\mathrm{STr}\ln(1 - Z\tilde{Z}) + \mathrm{STr}\ln(1 - w_+\mathcal{B}Z\mathcal{B}^\dagger w_-\tilde{Z})$$

The factors $w_\pm$ carry the difference in wave numbers in the advanced and the retarded Green functions and are irrelevant for what follows. Will be suppressed. So action is

$$\mathcal{A}(Z,\tilde{Z}) = -\mathrm{STr}\ln(1 - Z\tilde{Z}) + \mathrm{STr}\ln(1 - \mathcal{B}Z\mathcal{B}^\dagger\tilde{Z})$$

Exponential is multiplied by the "source terms"

$$\frac{\pi^2}{B^2}\Big(\mathrm{STr}[\sigma_3^s(1 - Z\tilde{Z}))^{-1}Z\tilde{Z}]\ \mathrm{STr}[\sigma_3^s(1 - \tilde{Z}Z)^{-1}\tilde{Z}Z] \\ +\mathrm{STr}[\sigma_3^s\tilde{Z}(1 - Z\tilde{Z}))^{-1}\sigma_3^s Z(1 - \tilde{Z}Z)^{-1}]\Big)$$

Here σ_3^s is the third Pauli spin matrix in superspace and breaks supersymmetry. Entire information on graph dynamics located in matrix $\mathcal{B}$.

3. Coset Space

Suppress indices (b, d) and rewrite action identically as

$$\mathcal{A} = -\mathrm{STr}\ln\frac{2}{1+Q\Lambda} + \mathrm{STr}\ln\left(1 - M\frac{Q\Lambda-1}{Q\Lambda+1}\right)$$

where

$$Q = g(Z)\Lambda g^{-1}(Z)$$

is Efetov's Q-matrix and where in retarded-advanced notation

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad M = \begin{pmatrix} \mathcal{B} & 0 \\ 0 & \mathcal{B}^\dagger \end{pmatrix},$$

$$g(Z) = \begin{pmatrix} (1-Z\tilde{Z})^{-1/2} & 0 \\ 0 & (1-\tilde{Z}Z)^{-1/2} \end{pmatrix}\begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix}.$$

The matrix Q remains unchanged if we replace $g \to gk$ provided that $[k,\Lambda] = 0$. Therefore, Q and $\mathcal{A}$ are defined in a coset space G/K with fundamental form $Q = g\Lambda g^{-1} = gk\Lambda k^{-1}g^{-1}$. We read k as gauge transformation. With $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ we have

$$Z = B(g)D^{-1}(g) = B(gk)D^{-1}(gk)$$

$$\tilde{Z} = C(g)A^{-1}(g) = C(gk)A^{-1}(gk)$$

as gauge-invariant coordinates of Q.

4. Universal Mode and Massive Modes

Universal Mode: Consider group element g_0 that in directed-bond space is multiple of unit matrix. Associated Q-matrix is $Q = g_0 \Lambda g_0^{-1}$. The gauge-invariant coordinates

$$Y = B_0 D_0^{-1} \ , \ \tilde{Y} = C_0 A_0^{-1}$$

define the universal mode. Define massive modes $(\zeta_{bd}, \tilde{\zeta}_{bd})$ by expanding $(Z_{bd}, \tilde{Z}_{bd})$ around $(Y, \tilde{Y})$, respecting coset structure. Write $(Z_{bd}, \tilde{Z}_{bd})$ as coordinates of $g_0 g(\zeta_{bd})$, replace $(\zeta_{bd}, \tilde{\zeta}_{bd})$ by gauge-invariant variables$(\xi_{bd}, \tilde{\xi}_{bd})$, find $Z_{bd} = (Y + \xi_{bd})(1 + \tilde{Y}\xi_{bd})^{-1}$, $\tilde{Z}_{bd} = (\tilde{Y} + \tilde{\xi}_{bd})(1 + Y\tilde{\xi}_{bd})^{-1}$. Expand in powers of $(\xi_{bd}, \tilde{\xi}_{bd})$.

Mathematically, linearization in $(\zeta_{bd}, \tilde{\zeta}_{bd})$ means that we consider the vector bundle of tangent vector spaces over G/K. For the 2 B variables $(\zeta_{bd}, \tilde{\zeta}_{bd})$ the approximation is that all 2 B tangent vector spaces together form a linear space. Justified if massive modes provide small corrections (size of gap).

To retain the correct number of independent variables we impose the constraints $\sum_{bd} \xi_{bd} = 0 = \sum_{bd} \tilde{\xi}_{bd}$.

5. Effective Action

We derive an invariance property of the action. That allows us to show that under the transformation $Z_{bd} \to (Y, \xi_{bd})$ the action takes the form

$$\mathcal{A}(Z,\tilde{Z}) \approx \mathcal{A}_{\rm bare} - \tfrac{2i\pi\kappa}{\Delta}\,{\rm STr}_s\,\tfrac{1}{1-Y\tilde{Y}}$$

where last term has standard form and where

$$\mathcal{A}_{\rm bare} = -{\rm STr}\ln(1-\xi\tilde{\xi}) + {\rm STr}\ln(1-\mathcal{B}\xi\mathcal{B}^{\dagger}\tilde{\xi})$$

For the source terms define $Q(\xi,\tilde{\xi}) = g(\xi)\Lambda g^{-1}(\xi)$. The contribution due to the massive modes is

$$\begin{aligned} &\tfrac{\pi^2}{B^2}\textstyle\sum_{bd}{\rm STr}_s\Big(\Sigma(Y,\tilde{Y})[Q(\xi_{bd},\tilde{\xi}_{bd})-1]\Sigma'(Y,\tilde{Y})[Q(\xi_{bd},\tilde{\xi}_{bd})-1]\Big)\\ &\quad+\tfrac{\pi^2}{B^2}\Big\{\textstyle\sum_{bd}{\rm STr}_s\Big(\Sigma(Y,\tilde{Y})[Q(\xi_{bd},\tilde{\xi}_{bd})-1]\Big)\\ &\qquad\qquad\times\textstyle\sum_{b'd'}{\rm STr}_s\Big(\Sigma'(Y,\tilde{Y})[Q(\xi_{b'd'},\tilde{\xi}_{b'd'})-1]\Big)\Big\}\end{aligned}$$

Clear separation of contributions due to universal mode and due to massive modes. To show that all integrals vanish for $B \to \infty$.

6. Evaluation

6.1 Simplify source terms by variable transformation. For instance write

$$\xi = \psi(1+\tilde{\psi}\psi)^{-1/2} \ , \ \tilde{\xi} = \tilde{\psi}(1+\psi\tilde{\psi})^{-1/2}$$

to transform $\xi\tilde{\xi}(1-\xi\tilde{\xi})^{-1} \to \psi\tilde{\psi}$ and $\tilde{\xi}\xi(1-\tilde{\xi}\xi)^{-1} \to \tilde{\psi}\psi$. Source terms contain only $\psi_{bd}\tilde{\psi}_{bd}$ or $\tilde{\psi}_{bd}\psi_{bd}$. Effective action becomes more complicated.

6.2 Expand effective action in powers of ψ_{bd} and $\tilde{\psi}_{bd}$. Keep in exponent only terms up to second order. Yields

$$\mathcal{A} \to \sum_{bd,b'd'} \mathrm{STr}_s[\psi_{bd}(\delta_{bb'}\delta_{dd'} - \mathcal{F}_{bd,b'd'})\tilde{\psi}_{b'd'}] \ .$$

Defines Gaussian superintegrals. Leading eigenvalue of $\mathcal{F}$ is + 1 and does not contribute. Projector onto space spanned by eigenvectors to the remaining eigenvalues is $\mathcal{P}$. Propagator is $\mathcal{P}(1-\mathcal{F})\mathcal{P}$. Gaussian integrals exist because of cutoff in spectrum, hence the term “Massive Modes”.

6.3 Expand exponential of higher order terms in $\mathcal{A}$ in Taylor series. To show that combination of every one of resulting terms with source terms vanishes for $B \to \infty$.

6.4 Estimates in terms of mean values. We need to estimate the dependence of all terms generated by the expansions on the dimension (2 B) of directed bond space for large B. Examples of mean values based on completeness:

$$\langle bd|\mathcal{P}(1-\mathcal{F})^{-1}\mathcal{P}|bd\rangle \approx \frac{1}{2B}\sum_{bd}\langle bd|\mathcal{P}(1-\mathcal{F})^{-1}\mathcal{P}|bd\rangle = \frac{1}{2B}\sum_{i\geq 2}\frac{1}{1-\lambda_i} \leq \frac{1}{a} \ .$$

$$\prod_{i=1}^{n}\langle b_i d_i|\mathcal{P}(1-\mathcal{F})^{-1}\mathcal{P}|b_{i+1}d_{i+1}\rangle \approx \frac{1}{a^n(2B)^{n-1}}\left(\delta_{b_{i+1}b_1}\delta_{d_{i+1}d_1} - \frac{1}{2B}\right) \ .$$

Unitarity of $\mathcal{B}$ implies $|\mathcal{B}_{bd,b'd'}| \approx \frac{1}{\sqrt{2B}}$. Every unrestricted summation over directed bond space is of order (2 B).

In that way we argue that all terms due to integration over massive modes vanish for $B \to \infty$. Mathematically unsatisfactory. Strict upper bounds needed.

6.5 Mathematically viable estimates: Estimate individual terms by establishing strict upper bounds rather than mean values. For instance, we have strictly

$$|\langle bd|\mathcal{P}(1-\mathcal{F})^{-1}\mathcal{P}|b'd'\rangle| \leq \frac{1}{a} \ .$$

So far, that approach has failed. Terms originating from expansion of

$$\mathrm{STr}\ln(1-\mathcal{B}\xi\mathcal{B}^{\dagger}\tilde{\xi})$$

cannot be shown to vanish individually for $B\to\infty$. Reason: Terms stemming from $\mathrm{STr}\ln(1-\xi\tilde{\xi})$ and from $\mathrm{STr}\ln(1-\mathcal{B}\xi\mathcal{B}^{\dagger}\tilde{\xi})$ are related by Ward identity and should be treated together. Work on that problem is in progress.

7. Conclusions

Study of two-point function of chaotic connected simple quantum graphs.

Unitary symmetry, incommensurate bond lengths, spectrum of Frobenius-Perron operator has finite gap.

Phase average of generating function done using supersymmetry and color-flavor transformation.

Effective action defined in coset space. Separate universal mode and massive modes. Linearize the latter.

Expand effective action up to second order to generate Gaussian superintegrals over massive modes. Taylor-expand remaining terms.

Averages based on mean values show that every term in the series so generated vanishes for $B \to \infty$. But strict proof of universality requires strict upper bounds. Work on that in progress.