Integrable structure at the integer quantum Hall plateau transition

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@ Yad Hashmona, ISF Workshop

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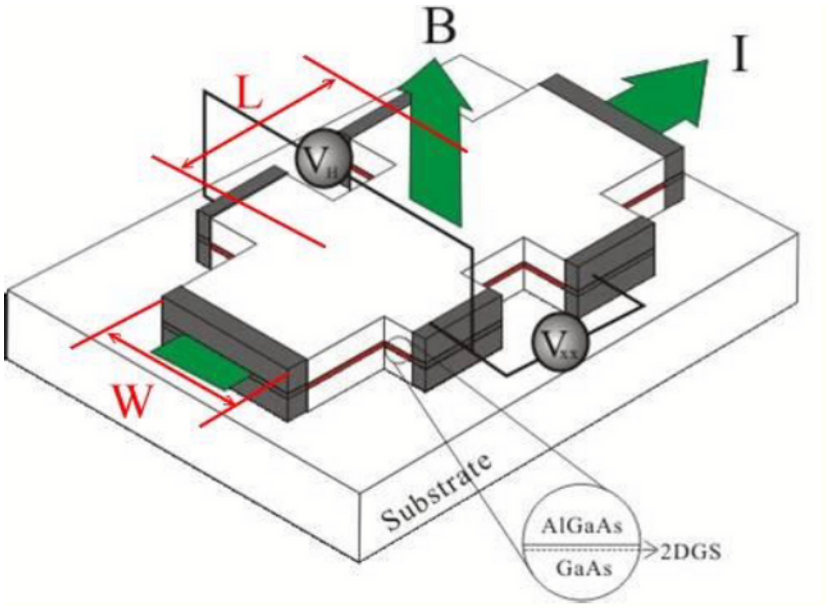
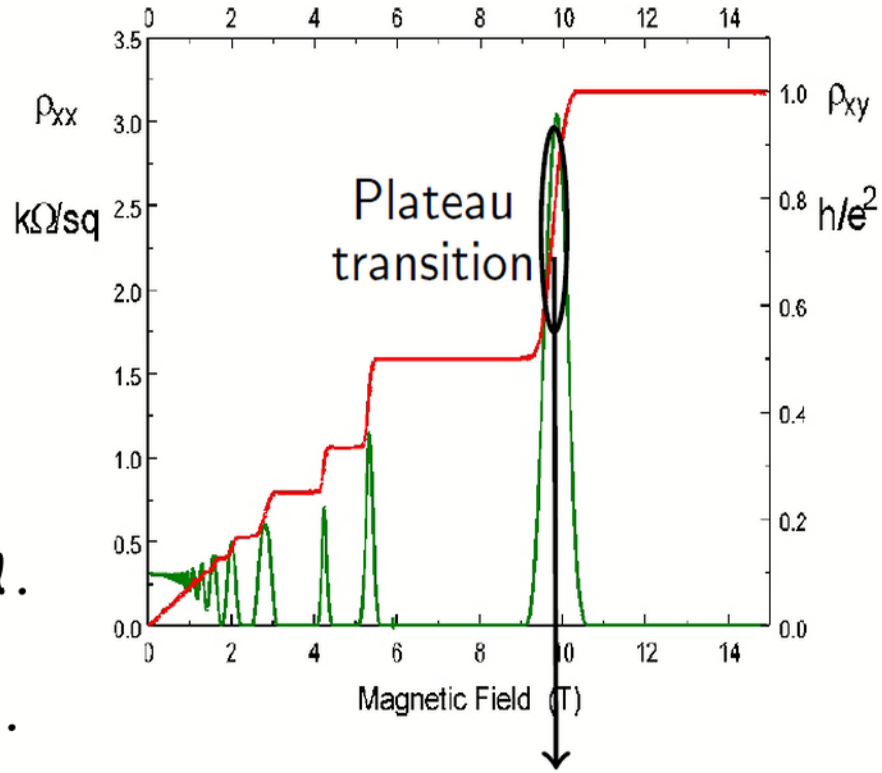
arXiv:1805.12555

based on R. Bondesan, D. Wieczorek & MZ: NPB (2017), PRL (2014)

Integer Quantum Hall Effect





Two-dimensional disordered electron gas at low temperature and in a strong magnetic field.

Hall resistance exhibits plateaus: $R_H = \frac{h}{ne^2}$.

Transition between plateaus is a critical phenomenon (of Anderson-localization type).
Could/should be a paradigm, but is not understood in quantitative detail...

Nonlinear sigma model (Pruisken et al., 1983)

weak localization

$$\mathcal{L} = \frac{\sigma_{xx}}{8} \mathrm{STr}\, \partial_\mu Q\, \partial_\mu Q + \frac{\sigma_{xy}}{8} \epsilon_{\mu\nu} \mathrm{STr}\, Q\, \partial_\mu Q\, \partial_\nu Q$$

θ-term

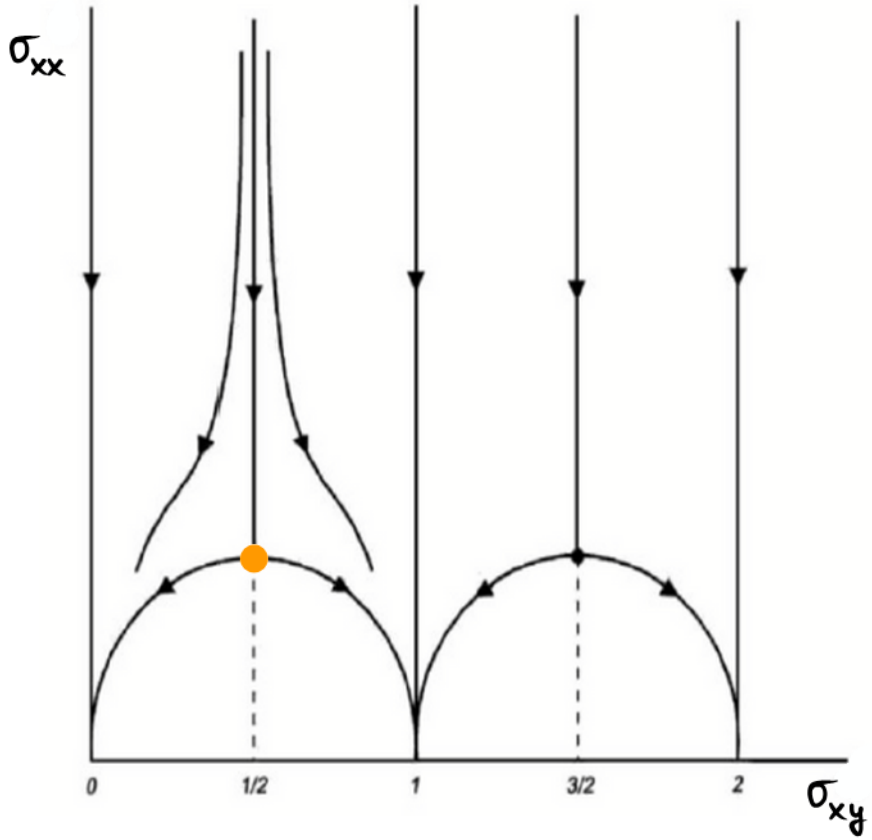
Wegner–Efetov SUSY method ⤳ target space

$$Q = u\, \Sigma_3\, u^{-1}, \qquad \Sigma_3 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$$

is complex Grassmann manifold $\mathcal{U}/K$

with global symmetry group $\mathcal{U} = U(r,r|2r)$

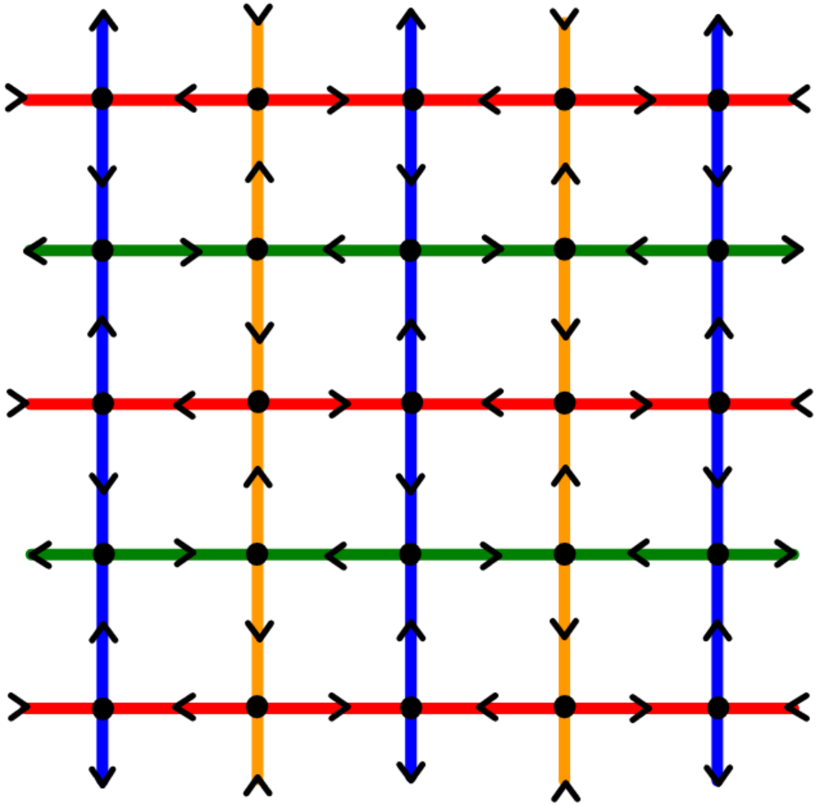
	ν	y
Huckestein & Kramer (1990)	2.34 ± 0.04	
Huckestein (1994)		-0.38 ± 0.04
Slevin & Ohtsuki (2009)	2.59 ± 0.005	-0.17 ± 0.04
Obuse, Gruzberg & Evers (2012)	2.62 ± 0.06	$-0.7 < y < 0$



Pruisken–Khmelnitskii RG flow diagram (conjectured, 1983)

CFT description?

Chalker-Coddington network model



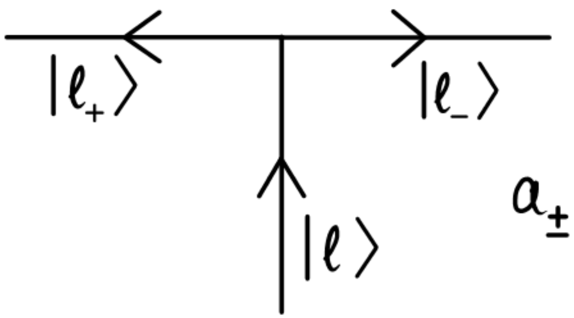
Hilbert space $= \mathbb{C}^{\#\text{links}}$

discrete-time evolution: $\psi_{t+1} = \mathcal{U}\,\psi_t$

$\mathcal{U} = \mathcal{U}_r\,\mathcal{U}_s$

$\mathcal{U}_s$ deterministic scattering at nodes:

$$\mathcal{U}_s\,|\ell\rangle = |\ell_+\rangle\,a_+ + |\ell_-\rangle\,a_-$$



$a_\pm = e^{\pm i\pi/4}/\sqrt{2}$

(at criticality)

Kac-Ward amplitudes

$\mathcal{U}_r$ diagonal in link basis:

$$\mathcal{U}_r\,|\ell\rangle = |\ell\rangle\,e^{i\phi(\ell)}$$

$\phi(\ell)$ uncorrelated random phases, uniformly distributed.

Note: $\exists$ conserved current

Nonlinear sigma model: Noether current

Now since Pruisken's non-linear sigma model enjoys invariance under the global symmetry group $U = \mathrm{U}(r, r|2r)$, it has a Noether current:

$$\partial^\nu J_\nu = 0\,, \quad J_\nu = u \begin{pmatrix} 0 & (u^{-1}\partial_\nu u)_{+-} \\ (u^{-1}\partial_\nu u)_{-+} & 0 \end{pmatrix} u^{-1}.$$

An equivalent formulation of that conservation law is

$$\boxed{\bar\partial J - \partial \bar J = 0\,,} \quad J = u \begin{pmatrix} 0 & j_{+-} \\ j_{-+} & 0 \end{pmatrix} u^{-1}, \quad \bar J = u \begin{pmatrix} 0 & \bar j_{+-} \\ \bar j_{-+} & 0 \end{pmatrix} u^{-1}.$$

Notation: $d = \partial + \bar\partial$, $\partial = dz \wedge \frac{\partial}{\partial z}$, $j = u^{-1}\partial u$, $\bar j = u^{-1}\bar\partial u$.

Hodge star operator $*$

$J_\nu * dx^\nu = J + \bar J$, $*J = -iJ$.

Affleck's argument (PRL 1985).

Current one-form $= J_\mu * dx^\mu$.

Current-current correlation function in the plane:

$$\langle J_\mu(x) J_\nu(x') \rangle = C_{\mu\nu}(x-x').$$

At the critical point, expect rotational invariance:

$$C_{\mu\nu}(x) = x_\mu x_\nu f(|x|^2) + \delta_{\mu\nu} |x|^2 g(|x|^2).$$

Scale invariance $\rightsquigarrow$ $C_{\mu\nu}(x) = \dfrac{x_\mu x_\nu f_0 + \delta_{\mu\nu}|x|^2 g_0}{|x|^4}$.

Current conservation $(\partial^\mu C_{\mu\nu} = 0)$ $\rightsquigarrow$ $C_{\mu\nu}(x) = \dfrac{x_\mu x_\nu - \delta_{\mu\nu}|x|^2/2}{|x|^4} f_0$.

Make decomposition $J_\mu * dx^\mu = J + \bar{J} = j^{10} * dz + j^{01} * d\bar{z}$.

It follows that $\langle j^{10}(z,\bar{z})\, j^{10}(w,\bar{w}) \rangle = \dfrac{n}{(z-w)^2}$,

$$\langle j^{01}(z,\bar{z})\, j^{01}(w,\bar{w}) \rangle = \frac{n}{(\bar{z}-\bar{w})^2},$$

and $\langle j^{10}(z,\bar{z})\, j^{01}(w,\bar{w}) \rangle = 0$.

Symmetry doubling

In a unitary CFT one infers

$$\frac{\partial}{\partial \bar{z}} \left\langle j^{10}(z,\bar{z})\, j^{10}(w,\bar{w}) \right\rangle = 0 \implies \frac{\partial}{\partial \bar{z}} j^{10} = 0 .$$

Hence $\bar{\partial} J = 0 = \partial \bar{J}$ Notation: $J = j^{10} * dz$, $\bar{J} = j^{01} * d\bar{z}$.

Symmetry group G doubles to $G_L \times G_R$ (Affleck, 1985).

Example.

antiferromagnetic quantum spin chain, $S = 1/2$.

	target space
— non-linear sigma model at $\theta = \pi$ (Haldane 1981-83)	$SU(2)/U(1)$
— Wess–Zumino–Witten model (Affleck 1985)	$SU(2)$

The symmetry group $SU(2)$ doubles to $SU(2)_L \times SU(2)_R$.

Wess-Zumino-Witten model

Lagrangian formulation. WZW field $M : \Sigma \to X$. Action functional:

$$S_n^{WZW}[M] = \frac{n}{4\pi i} \int_\Sigma \left(\mathrm{Tr}\, M^{-1}\partial M \wedge M^{-1}\bar{\partial} M + \tfrac{1}{3} d^{-1} \mathrm{Tr}\, (M^{-1} dM)^{\wedge 3} \right)$$

Left and right translations $M \mapsto u_L M u_R^{-1}$ $(u_L, u_R \in \mathcal{U})$ give rise to currents

$$J = \partial M \cdot M^{-1} \quad \text{holomorphic}$$

$$\bar{J} = M^{-1} \bar{\partial} M \quad \text{anti-holom.}$$

$$\boxed{\bar{\partial} J = 0 = \partial \bar{J}} \quad \checkmark$$

Operator formalism.

Currents $J^A = \mathrm{Tr}\,(A \partial M \cdot M^{-1})$, $\bar{J}^A = \mathrm{Tr}\,(A M^{-1} \bar{\partial} M)$.

Current algebra $\hat{\mathfrak{u}}$. Operator product expansion:

$$J^A(z)\, J^B(w) \sim \mathrm{Tr}(AB) \frac{n}{(z-w)^2} + \frac{J^{[A,B]}(w)}{z-w}$$

The Current Algebra Conundrum

- Symmetry group $\mathcal{U} = U(r, r|2r)$ must act (by conjugation) on target X. All known CFTs with continuous symmetries are Wess-Zumino-Witten models or Goddard-Kent-Olive coset theories (= gauged WZW models).

- WZW models (with non-compact target X) are ruled out by Chamon, Mudry & Wen (1995)
  - RG-instability against infinity of relevant $\mathcal{U}$-invt perturbations
  - exact results for related critical point (class C) Read & Saleur (2001)

- Coset theories X/H with $H \subset \mathcal{U}$ greater than center of $\mathcal{U}$ are ruled out by $\mathcal{U}$-symmetry. Gauging by center $H = U(1)$ does not remove RG instability.

- From NLσM expect $\langle J^{+-} J^{-+} \rangle$ and $\langle J^{++} J^{--} \rangle$ to be critical, but according to basic principles $\langle J^{++} J^{++} \rangle$ and $\langle J^{--} J^{--} \rangle$ must be trivial.
  ⇝ current algebra with degenerate bilinear form ?

Observation: Affleck's argument does apply to the $\mathcal{U}(1)$ conserved current of the CC model.

Proposed Resolution (arXiv:1805.12555)

Some kind of symmetry breaking takes place:

only a subset of the $U(r,r|2r)$ Noether currents (the "physical" ones) are promoted to (anti-)holomorphic currents. The algebra of critical currents is an affine Lie superalgebra $\widehat{gl}(r|r)_{n,\gamma}$ defined by the operator product expansion (OPE)

$$J^X(z)J^Y(w) \sim \frac{\langle X, Y\rangle_{n,\gamma}}{(z-w)^2} + \frac{J^{[X,Y]}(w)}{z-w}$$

with invariant bilinear form deformed by a parameter γ:

$$\langle X, Y\rangle_{n,\gamma} = -n\,\mathrm{STr}(XY) + \gamma\,\mathrm{STr}(X)\,\mathrm{STr}(Y)$$

The energy-momentum tensor

$$T_{\widehat{\mathfrak{gl}}(r|r)_{n,\gamma}} = -\frac{(-1)^{|\beta|}}{2n}(J_\alpha^{\ \beta} J_\beta^{\ \alpha}) + \frac{1-\gamma}{2n^2}(J_\alpha^{\ \alpha} J_\beta^{\ \beta})$$

satisfies the OPE for the Virasoro algebra with conformal charge $c=0$.

The observed phenomenology is obtained for $n=4$ and $\gamma=1$.

Functional integral realization

by $GL(r|r)_{n,\gamma}$ WZW model:

$$S^{\rm WZW}_{n,\gamma}[M] = S^{\rm WZW}_{n}[M] - \frac{\mathrm{i}\,\gamma}{2\pi}\int \mathrm{STr}(M^{-1}\partial M)\wedge \mathrm{STr}(M^{-1}\bar\partial M)$$

target = Riemannian symmetric superspace of type A|A

base manifold $\mathrm{Herm}^+(r) \times U(r)$

OPE of the fundamental field with the holomorphic current:

$$J_\alpha{}^\beta(z)\, M_\gamma{}^\delta(w,\bar w) \sim \delta^\beta_\gamma\, \frac{M_\alpha{}^\delta(w,\bar w)}{z-w}$$

and with the holomorphic energy-momentum tensor:

$$T(z)M(w,\bar w) \sim h\,\frac{M(w,\bar w)}{(z-w)^2} + \frac{\partial M(w,\bar w)}{z-w} \quad , \quad h = \frac{1-\gamma}{2n^2} \, .$$

Note: the conformal weight h vanishes for $\gamma = 1$.

Predictions

Current-current correlation function:

$$\langle J_\mu^{+-}(x)\, J_\nu^{-+}(y)\rangle = \nabla_{x,\mu}\nabla_{y,\nu}\, \mathbb{E}\left(\left|\langle x|(1-U_\epsilon)^{-1}|y\rangle\right|^2\right)$$

$$\langle J_\mu^{++}(x)\, J_\nu^{--}(y)\rangle = \nabla_{x,\mu}\nabla_{y,\nu}\, \mathbb{E}\left(\langle x|(1-U_\epsilon)^{-1}|x\rangle\, \overline{\langle y|(1-U_\epsilon)^{-1}|y\rangle}\right).$$

- Current algebra $\widehat{gl}(r|r)_{n,\gamma}$ predicts amplitude ratio

$$\langle J_\mu^{++}(x)\, J_\nu^{--}(y)\rangle = -\frac{\gamma}{n}\,\langle J_\mu^{+-}(x)\, J_\nu^{-+}(y)\rangle.$$ NEW!

- Mean conductance (cylinder geometry): $\mathbb{E}(G) = \frac{n}{2\pi} = 0.638$

Multi-fractal scaling exponents Δ_q.

Positive field component $M_0{}^0 \equiv e^\varphi > 0$ gives $(M_0{}^0)^q(r) \longleftrightarrow |\psi_c(r)|^{2q}$

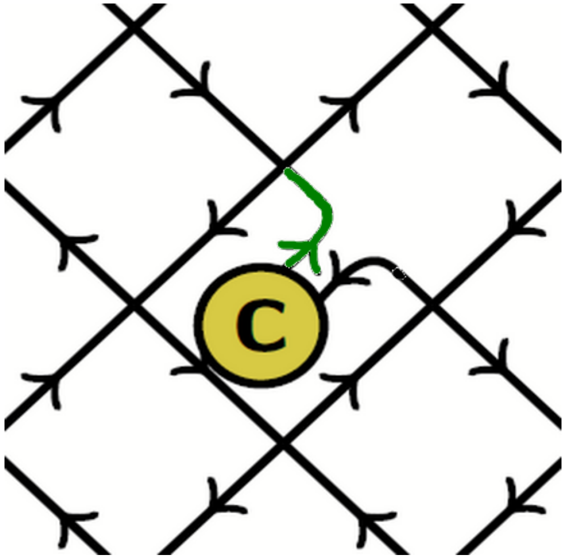
Now $\quad T(z)(M_0{}^0)^q(w,\bar{w}) \sim \dfrac{h_q}{(z-w)^2}(M_0{}^0)^q(w,\bar{w}) + \ldots$

with $\quad h_q = -\dfrac{q(q-1)}{2n} + (1-\gamma)\dfrac{q^2}{2n^2} \quad \overset{\gamma=1}{\Longrightarrow} \quad \Delta_q = h_q + \bar{h}_q = \dfrac{q(1-q)}{n}$

Note: free field correlation functions at $n=1$ and $\gamma=0$.

Statistics of wave intensities

Bondesan, Wieczorek & MZ
PRL (2014), NPB-FS (2017)

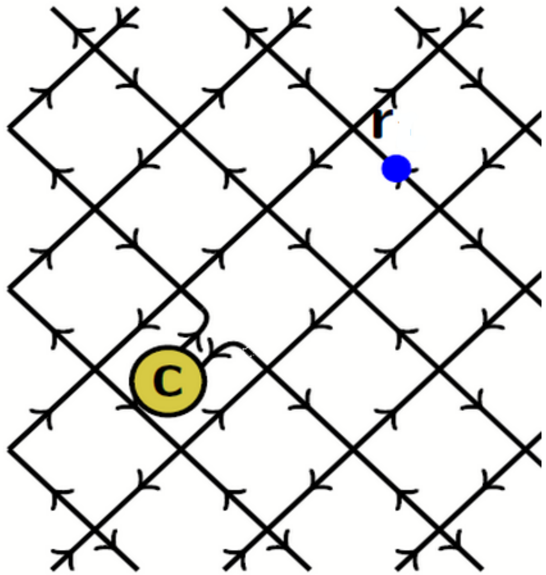


$\psi_c = U\psi_c$ stationary "scattering" state (quasi-energy zero) for incoming-wave boundary conditions $\psi_c(c) = 1$.

↳ Observable: $|\psi_c(r)|^2$ for large distances $|r-c|$

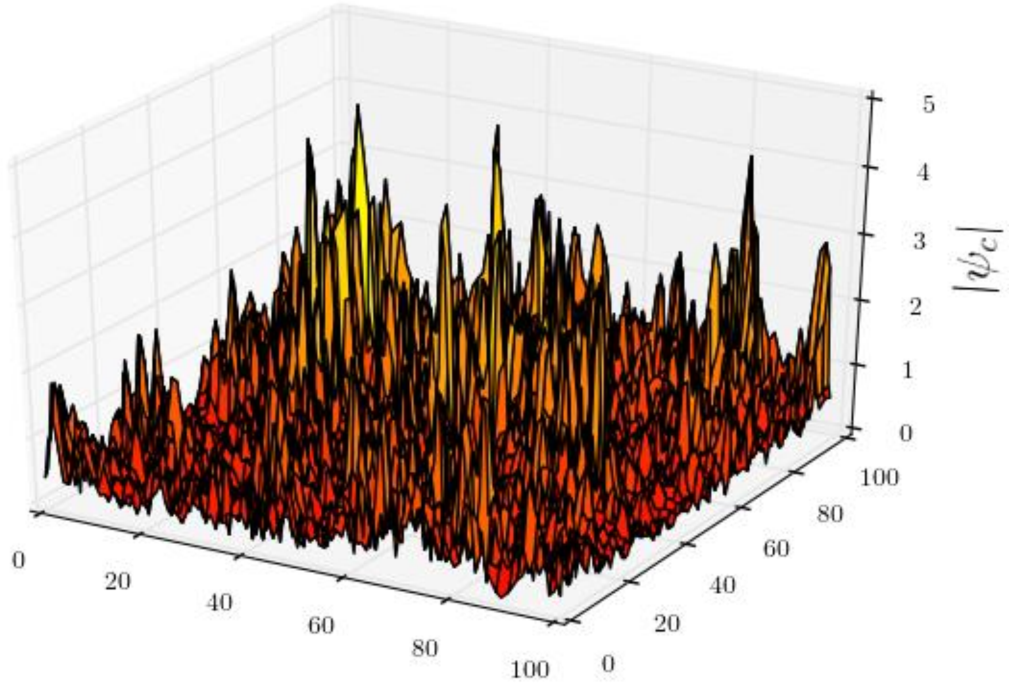
Prediction from Abelian OPE, crossing symmetry of 4-point fctn:

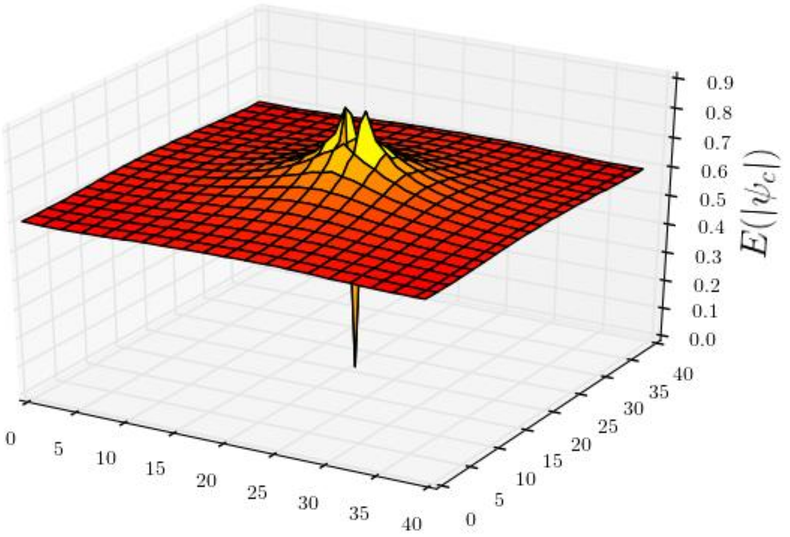
$$\mathbb{E}\left(|\psi_c(r)|^{2q}\right) \simeq |r-c|^{-2\Delta_q}, \quad \Delta_q = \frac{1}{n} q(1-q).$$

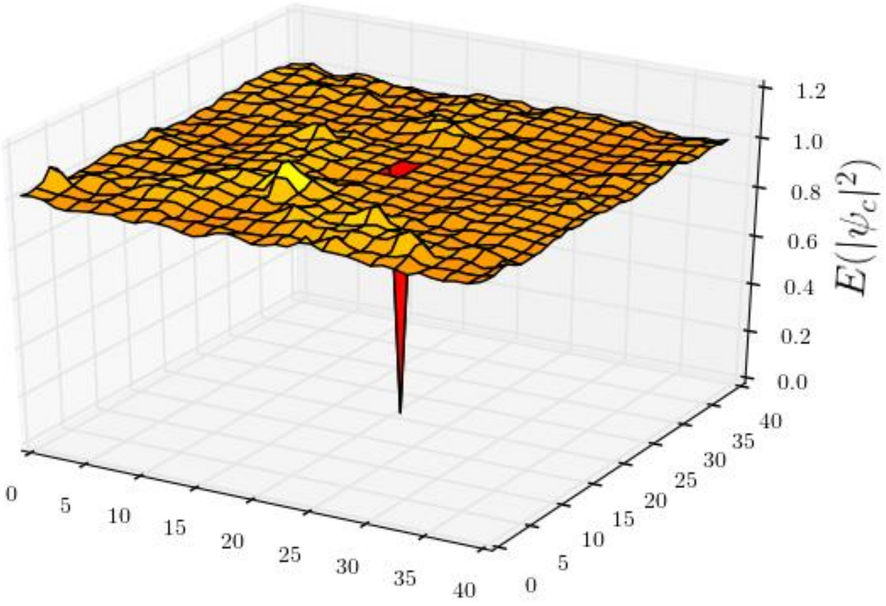


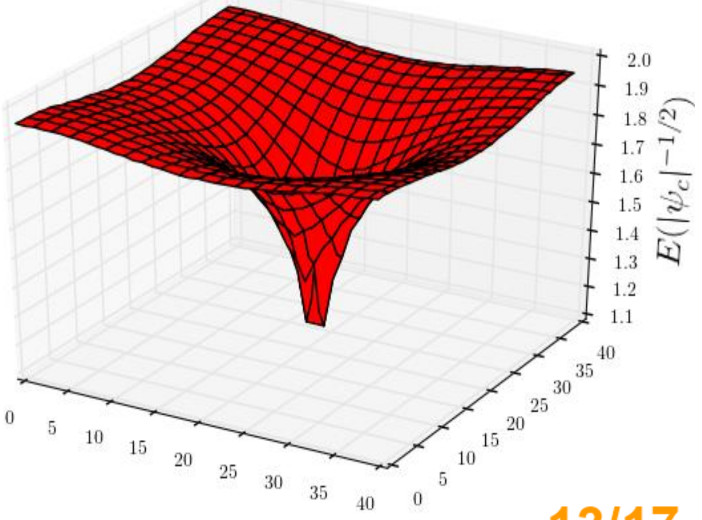
Interpretation: $\log|\psi_c(r)|^2 \equiv \varphi(r)$

GFF with background charge $Q=1$, stiffness $\frac{1}{n}$. Numerical simulations gave $\frac{1}{n} \cong 0.26$ (... 0.28)


$|\psi_c|$


$E(|\psi_c|)$


$E(|\psi_c|^2)$


$E(|\psi_c|^{-1/2})$

CC network model $\to$ SUSY vertex model

A variant (due to N. Read of the Wegner–Efetov supersymmetry method trades the task of taking disorder averages for a statistical mechanics problem of new (collective) variables that admit a continuum limit (at the critical point)

Uses second quantization on a Fock space for bosons and fermions.

Retarded sector: $\mathcal{U} = e^X$, $\mathrm{Re}\, X < 0$.

bosons : $\mathrm{Det}^{-1}_{\mathbb{C}^N}(1-\mathcal{U}) = \mathrm{Tr}_{S(\mathbb{C}^N)}\, \rho_B(\mathcal{U})$

fermions : $\mathrm{Det}(1-\mathcal{U}) = \mathrm{STr}\, \rho_F(\mathcal{U})$

where $\rho_B(e^X) = e^{b^\dagger X b}$, $\rho_F(e^X) = e^{f^\dagger X f}$

Advanced sector: $\mathcal{U} = e^X$, $\mathrm{Re}\, X > 0$.

$b^\dagger \to -b$, $b \to b^\dagger$,
$f^\dagger \to +f$, $f \to f^\dagger$.

SUSY vertex model

Use $\rho(\mathcal{U}_r \mathcal{U}_s) = \rho(\mathcal{U}_r)\,\rho(\mathcal{U}_s)$.

Random phase average projects Fock space to subspace of $U(1)$ singlets

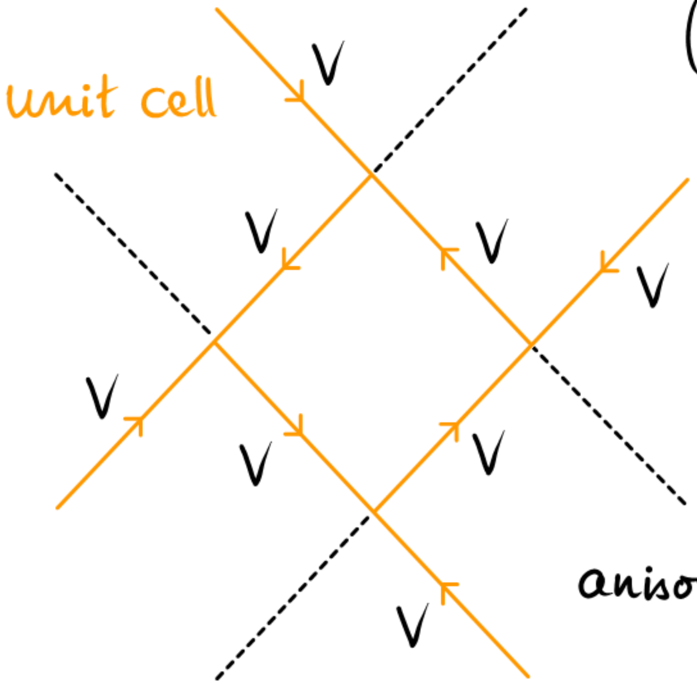
$\rightsquigarrow \quad \mathcal{H} = \bigotimes_{\text{links}} V(\ell)$ state space of SUSY vertex model

where $V = \bigoplus_{n_b^+ + n_f^+ = n_b^- + n_f^-} \left(S^{n_b^+}(\mathbb{C}) \otimes \Lambda^{n_f^+}(\mathbb{C}) \otimes S^{n_b^-}(\mathbb{C}^*) \otimes \Lambda^{n_f^-}(\mathbb{C}^*) \right)$

is irreducible highest-weight representation for $\mathcal{U} = U(1,1|2)$

(r replicas: $\mathcal{U} = U(r,r|2r)$

global symmetry of statistical sum).

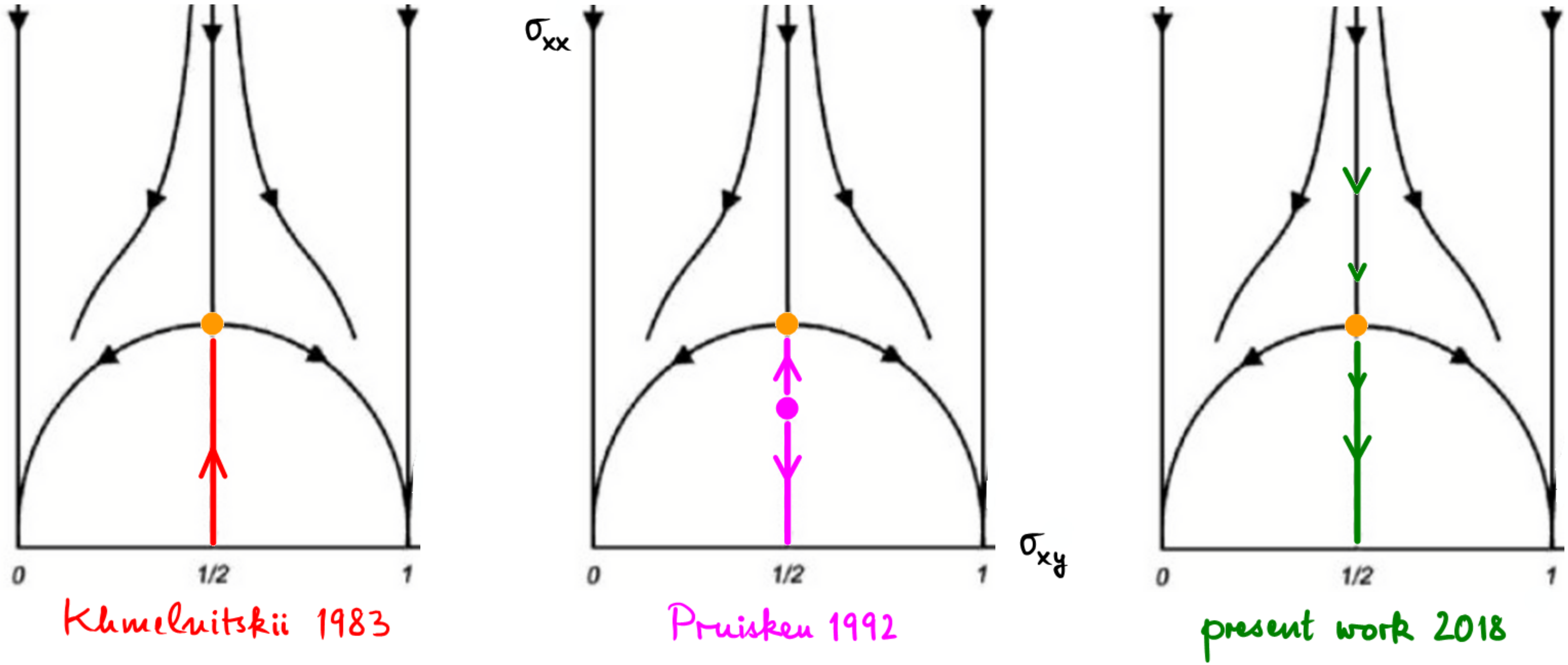


Q (NLσM) = coherent-state variable for V,

physical meaning: number of visits by Feynman path.

anisotropic limit $\rightsquigarrow$ "antiferromagnetic" (super)spin chain.

Qualitative picture of RG flow

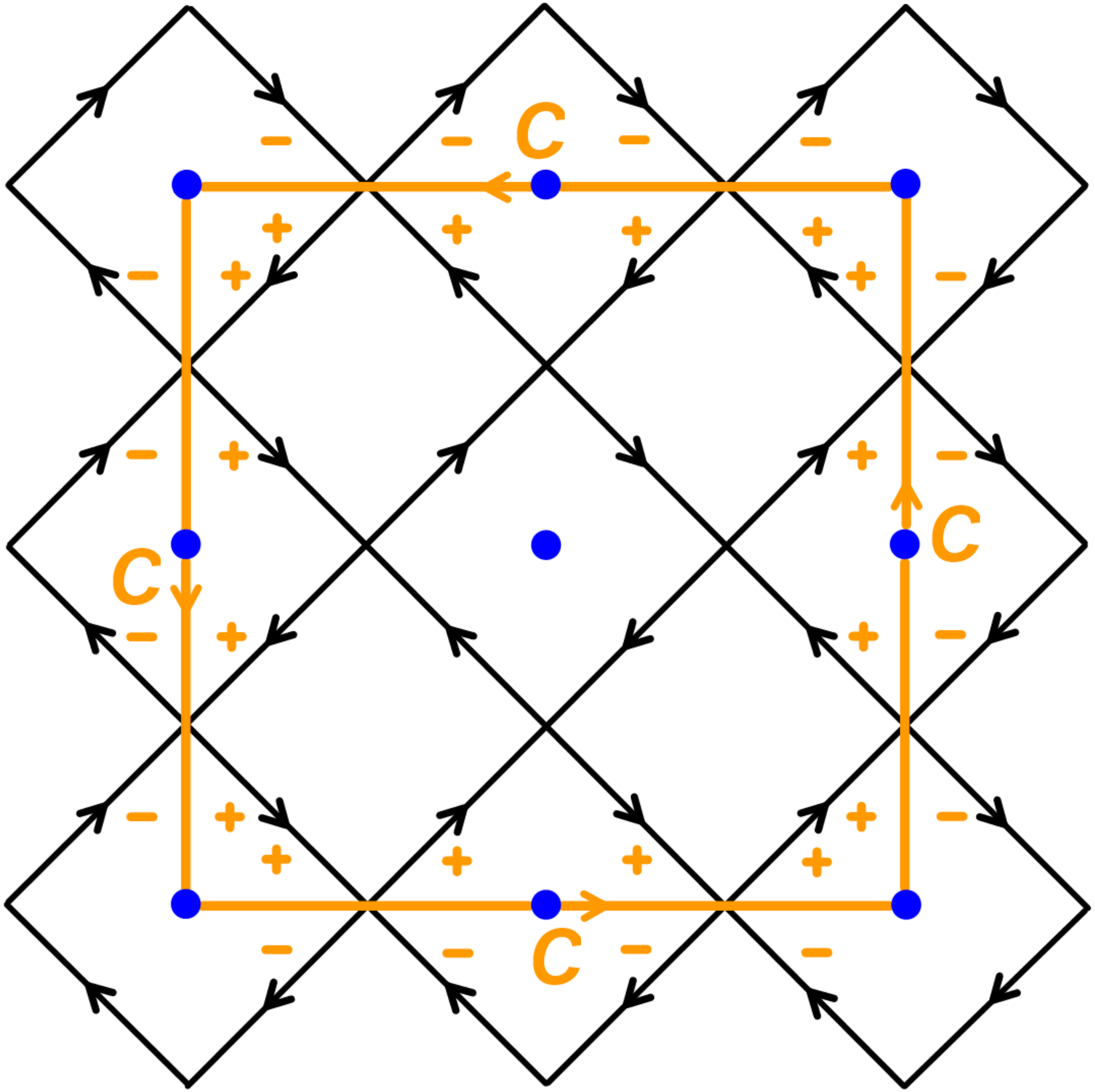


All symmetry-allowed CFT perturbations are marginal:

$$\int \mathrm{STr}\,\mathrm{Ad}(M) J \wedge \bar{J}, \quad \int \mathrm{STr} J \wedge \mathrm{STr} \bar{J}, \quad \int \mathrm{STr}\, \bar{\partial} J = - \int \mathrm{STr}\, \partial \bar{J} \, .$$

Open Problems

- quantitative analysis of marginal CFT perturbations
- other symmetry classes: spin quantum Hall effect (class C)
  thermal quantum Hall effect (class D)
- numerical checks of predictions from current algebra
- logarithmic CFT: atypical sector
- mathematical control of scaling limit — discrete holomorphicity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