

Painlevé' Transcendents
and their Appearance in Physics and
Random Matrices

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Yad Hashmonai (2009)

Painlevé Equations

I. $u_{xx} = 6u^2 + x$

II. $u_{xx} = xu + 2u^3 + \underline{a}$

III. $u_{xx} = \frac{1}{u}u_x^2 - \frac{u_x}{x} + \frac{1}{x}(\alpha u^2 + \beta) + \gamma u^3 + \frac{\delta}{u}$

IV. $u_{xx} = \frac{1}{2u}u_x^2 + \frac{3}{2}u^3 + 4xu^2 + 2(x^2 - \alpha)u + \frac{\beta}{u}$

V. $u_{xx} = \frac{3u - 1}{2u(u - 1)}u_x^2 - \frac{1}{x}u_x + \frac{(u - 1)^2}{x^2}(\alpha u + \frac{\beta}{u}) + \frac{\gamma u}{x} + \frac{\delta u(u + 1)}{u - 1}$

VI. $u_{xx} = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u - 1} + \frac{1}{u - x}\right)u_x^2 - \left(\frac{1}{x} + \frac{1}{x - 1} + \frac{1}{u - x}\right)u_x + \frac{u(u - 1)(u - x)}{x^2(x - 1)^2} \left(\alpha + \beta\frac{x}{u^2} + \gamma\frac{x - 1}{(u - 1)^2} + \delta\frac{x(x - 1)}{(u - x)^2}\right)$

($\nu, \alpha, \beta, \gamma, \delta$ are complex parameters)

P. Painlevé B. Gambier (1900, 1910).

. Introduction.

$$u_{xx} = F(u, u_x, \alpha) \oplus P.P.$$



Painlevé Equations

Painlevé Equations

P₁ - P₆

New
transcendents



Isomorphy deformations of

$$\frac{d\Psi}{d\lambda} = A(\lambda; \alpha)\Psi$$

• Riemann-Hilbert Problem

• Jost, Minami, Flaschka, Mada





The Riemann-Hilbert representation of

Painlevé' transcendents :

$$Y_+(x) = Y_-(x) G(x, \infty)$$

$$\text{DE} \Leftrightarrow x \in \Gamma$$

$$G(x, \infty) = e^{i\theta(x, \infty)} S e^{-i\theta(x, \infty)}$$

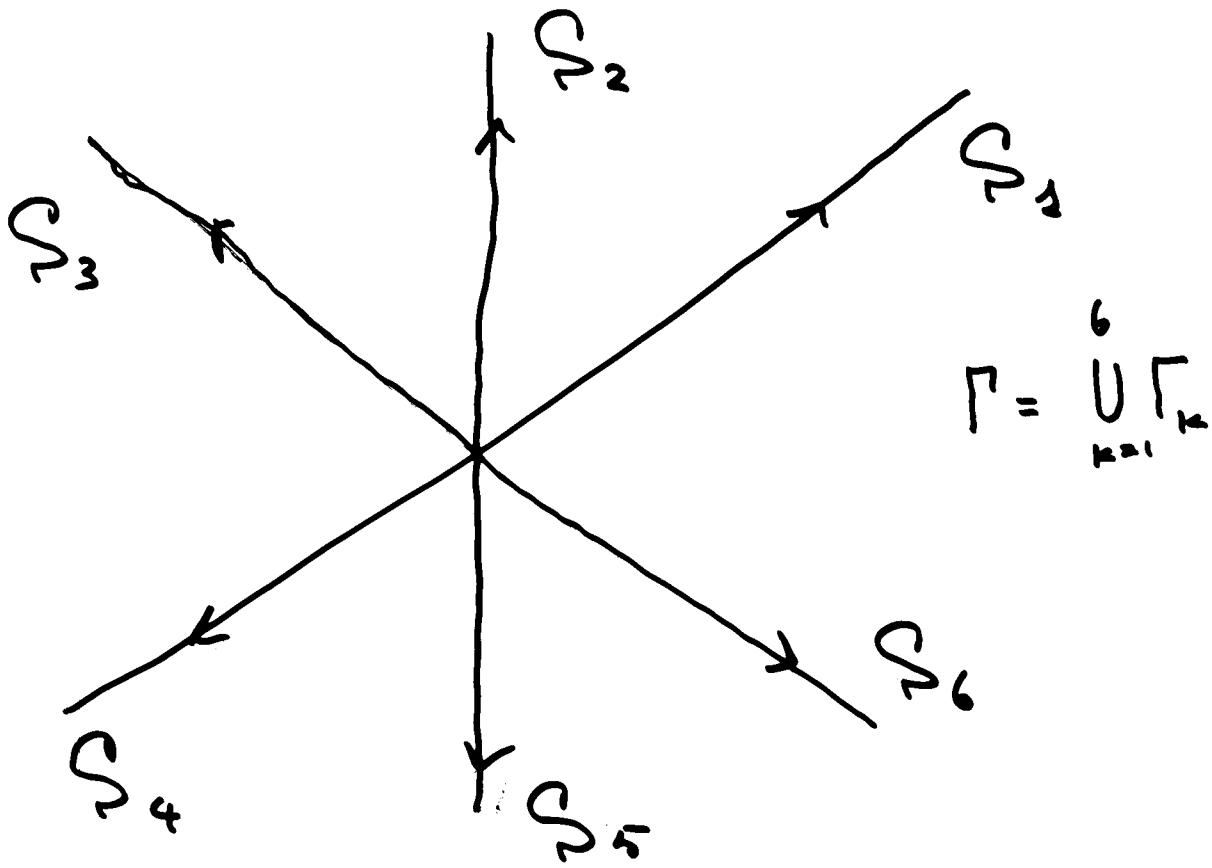
S - collection of Stokes matrices

Γ - collection of anti-Stokes rays

$$U(\infty) = \text{"res"}_{x=\infty} Y(x, \infty)$$

2'

Example. P $\overline{\text{II}}$, $\alpha=0$.



$$S_1 = \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix}, S_2 = \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix}, \dots$$

$$s_4 = -s_1, s_5 = -s_2, s_6 = -s_3$$

$$s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0 \iff S_1 \dots S_6 = I$$

$$G_\kappa(\gamma) = e^{i\theta(\gamma)} S_\kappa e^{-i\theta(\gamma)} \quad \gamma \in \Gamma_\kappa$$

$$\theta(\gamma) = \theta(\gamma, \infty) = -\left(\frac{4}{3}\gamma^3 + \alpha\gamma\right) G_3$$

$$G_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\Upsilon(\gamma) \equiv \Upsilon(\gamma, \infty) :$

• $\Upsilon(\gamma) \in H(C \setminus \Gamma)$

• $\Upsilon_+(\gamma) = \Upsilon_-(\gamma) G_\kappa(\gamma), \gamma \in \Gamma$

• $\Upsilon(\gamma) \mapsto I, \gamma \rightarrow \infty$

$$u(x) = 2(m_2)_{12}$$

$$Y(\lambda) \sim I + \frac{m_1}{\lambda} + \dots \quad \lambda \rightarrow \infty$$

$$u_{xx} = x u + 2u^3$$

Linear system:

$$\Psi(\lambda) := Y(\lambda) e^{i\theta(\lambda)}$$

Note:

$$\frac{d\Psi}{d\lambda} = A(\lambda) \Psi$$

$$i \frac{d\theta}{d\lambda} = \text{BNF of } A(\lambda)$$

also:

$$Z_2 A(-\lambda) Z_2 = -A(\lambda)$$

$$A(\lambda) = \lambda^2 A_2 + \lambda A_1 + A_0$$

$$Z_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -4i\lambda^2 - ix - 2iu^2 & i\lambda u - 2u_x \\ -i\lambda u - 2u_x & 4i\lambda^2 + ix + 2iu^2 \end{pmatrix}$$

The List of Painlevé' RH problems

$$\frac{d\Psi}{d\lambda} = A(\lambda)^{\frac{1}{2}} \quad , \quad \theta(\lambda, \infty)$$

$$\left(\frac{d\theta}{d\lambda} - \text{BNF of } A(\lambda) \right)$$

P I: $A(\lambda) = \sum_{k=-1}^4 \lambda^k A_k \quad \mathcal{Z}_1 A(-\lambda) \mathcal{Z}_1 = -A(\lambda)$

$$\theta(\lambda) = (\lambda^5 + \alpha \lambda) \mathcal{Z}_3$$

P II: $A(\lambda) = \sum_{k=-1}^2 \lambda^k A_k \quad \mathcal{Z}_2 A(-\lambda) \mathcal{Z}_2 = -A(\lambda)$

$$\theta(\lambda) = (\lambda^3 + \alpha \lambda) \mathcal{Z}_3$$

or

$$A(\lambda) = \sum_{k=0}^2 \lambda^k A_k \quad \theta(\lambda) = (\lambda^3 + \alpha \lambda + 2 \ln \lambda) \mathcal{Z}_3$$

P_{III} : $A(\lambda) = A_0 + \frac{A_{-1}}{\lambda} + \frac{A_{-2}}{\lambda^2}$

$$\theta(\lambda) = -\infty \left(\lambda + \frac{1}{\lambda} \right) z_2$$

P_{IV} : $A(\lambda) = \sum_{k=0}^3 A_k \lambda^k \quad z_2 A(-\lambda) b_3 = -A(\lambda)$

$$\theta(\lambda) = (\lambda^4 + \infty \lambda^2) z_3$$

or

$$A(\lambda) = A_3 \lambda + A_0 + \frac{A_{-1}}{\lambda} \quad \theta(\lambda) = (\lambda^2 + \infty \lambda) z_3$$

P_V : $A(\lambda) = A_0 + \frac{A_1}{\lambda} + \frac{A_2}{\lambda-1} \quad \theta(\lambda) = (\infty \lambda) z_3$

P_{VI} : $A(\lambda) = \frac{A_1}{\lambda} + \frac{A_2}{\lambda-1} + \frac{A_3}{\lambda-\infty}$

Garnier; Jimbo, Miwa; Flaschka, Newell

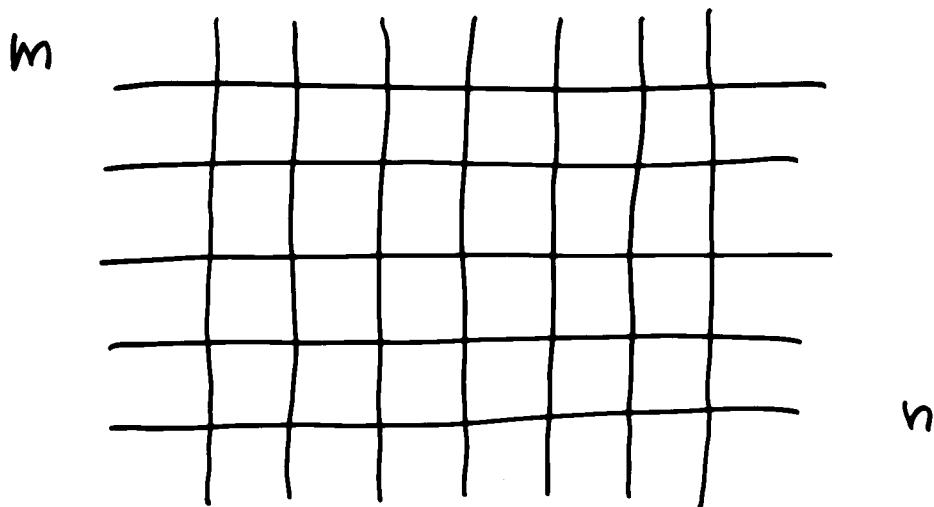
The principal point:

The RH representation \equiv non-abelian
analog of contour
integral representation.



The possibility of global asymptotic
analysis of the Painlevé' functions.

Ising Model



Energy Functional :

$$E(\beta) = -E^v \sum_{n,m=0}^L \beta_{mn} \beta_{m+1,n}$$

$$- E^h \sum_{n,m=0}^L \beta_{mn} \beta_{m,n+1}$$

$\beta_{mn} = \pm 1$ - value of spin variable
at (m,n) -site

Partition function:

$$Z_L = \sum_{\beta} e^{-\beta E(\beta)}, \quad \beta = \frac{1}{kT}$$

free energy:

$$f = -kT \lim_{L \rightarrow \infty} \frac{1}{L^2} \ln Z_L$$

two-point correlation function:

$$\langle \delta_{00} \delta_{MN} \rangle_L = \frac{1}{Z_L} \sum_{\beta} \delta_{00} \delta_{MN} e^{-\beta E(\beta)}$$

$$\boxed{\langle \delta_{00} \delta_{MN} \rangle = \lim_{L \rightarrow \infty} \langle \delta_{00} \delta_{MN} \rangle_L}$$

(1944) Onsager: Free energy, T_c :

$$\sinh \frac{2E^v}{kT_c} \sinh \frac{2E^h}{kT_c} = 1$$

$T = T_c$:

$$\langle \mathcal{Z}_{\infty} \mathcal{Z}_{NN} \rangle \sim A N^{-1/4} \quad A = 0.645\dots$$

$T > T_c$:

$$\langle \mathcal{Z}_{\infty} \mathcal{Z}_{NN} \rangle \sim \frac{1}{(\pi N)^{1/2}} \frac{k_s^N}{(1 - k_s^2)^{1/4}}$$

$$k_s = \sinh \frac{2E^v}{kT} \sinh \frac{2E^h}{kT}$$

Wu, McCoy, Tracy, Barouch (1976)

B. McCoy, The connection between SM & QFT

hep-th/9403094

Scaling Theory.

$$C_+(\tau) := \lim_{M_+^2} \frac{1}{M_+^2} \langle \mathcal{Z}_{00} \mathcal{Z}_{NN} \rangle$$

$N \rightarrow \infty, T \rightarrow T_c$

$$N(T-T_c) = O(1)$$

$$\tau = \frac{1}{2} N(T-T_c)$$

$$M_+ = \left[\left(\tanh^2 \frac{E}{kT} \tanh^2 \frac{E}{kT} \right)^{-1} \right]^{1/8} \approx (T-T_c)^{1/8}$$

$$C_+(\tau) = \frac{1}{2} (1 - \nu(\tau)) \gamma^{-1/2} (\tau)$$

$$= \exp \int_{\tau/2}^{\infty} \frac{1}{4} \frac{(1-\gamma^2)^2 - \gamma'^2}{\gamma^2} \propto d\gamma$$

$$\gamma \equiv \gamma(x) : \boxed{\gamma'' = \frac{1}{2} (\gamma')^2 - \frac{1}{x} \gamma' + \gamma^3 - \frac{1}{2}} - P\bar{III}.$$

$T > T_c$ asymptotic \Rightarrow B.C.

$$\gamma(x) \sim 1 - \frac{1}{\sqrt{\pi}} x^{-1/2} e^{-2x} \quad (1)$$

$x \rightarrow +\infty$

Key question:

$$(1) \Rightarrow C_+(\tau) \sim \text{const } \tau^{-1/4} !$$

? $\tau \rightarrow 0$



matching with $T = T_c$ asymptotics

McCoy, Tracy, Wu (1977)

Tracy (1991) \leftrightarrow the constant.

McCoy-Tracy-Wu connection formulae

$$\gamma \sim 1 - \gamma \sqrt{\pi} x^{-\frac{1}{2}} e^{-2x}, \quad x \rightarrow +\infty$$

$$|\gamma| \leq \frac{1}{\pi}$$

↓ !

$$\gamma \sim B x^{\beta}, \quad x \rightarrow 0, \quad |\gamma| < \frac{1}{\pi}$$

$$\beta = \frac{2}{\pi} \arcsin \pi \gamma, \quad B = 2^{-3\beta} \frac{\Gamma(\frac{4-\beta}{2})}{\Gamma(\frac{1+\beta}{2})}$$

$$\gamma \sim -\alpha \left(\ln \frac{x}{4} + \gamma_E \right), \quad x \rightarrow 0, \quad \alpha = \frac{1}{\pi}$$

Painlevé before large N limit.

$$N < \infty, \quad t := (\sinh^2 E''_{kT} \sinh^2 E''_{kT})^{-2}$$

$$\zeta(t) := t(t-1) \frac{d}{dt} \ln \langle z_0 z_{NN} \rangle - \frac{1}{4} t$$

$$(t(t-1) \frac{d^2 \zeta}{dt^2})^2 = N^2 \left[(t-1) \frac{d\zeta}{dt} - \zeta \right]^2$$

$$- 4 \frac{d\zeta}{dt} \left[(t-1) \frac{d\zeta}{dt} - \zeta - \frac{1}{4} \right] \left(t \frac{d\zeta}{dt} - \zeta \right)$$

PVI (Z-form) Jimbo, Miwa (1981)

⊕ Discrete PVI ($Z_N \rightarrow Z_{N+1}$),

! Toda for $Z_N(t)$

7'

More on Painleve' and Stat. Mechanics

XY spin- $\frac{1}{2}$ model, XX0, XXX, XXZ

Six-vertex model

see McCoy's review and

Palmer, Perk,

Forrester, Witte,

Izergin, Korepin, Slavnov, I.,

Kanzieper

Bleher, Fokin

1.

Self-similar reduction of integrable
PDEs.

$$V_t + V_{zzz} - 6V^2 V_z = 0 \quad - \text{mkdV equation}$$

put

$$V(t, z) = \frac{1}{(3t)^{1/3}} u\left(\frac{z}{(3t)^{1/3}}\right)$$

then $u \equiv u(\infty)$ satisfies $P_{II}^{\bar{1}}$:

$$u_{xx} = xu + 2u^3 + \text{const}$$

Ablowitz, Segur (1977)

(Zakharov, Manakov, Shabat)

Moreover, matching with the asymptotics

of $V(t, z)$ as $t \rightarrow \pm\infty$, $z/t = O(1)$,

the following 1-parameter family of $P_{II}^{\bar{1}}$
appear: (Ablowitz-Segur)

$$U(x) \equiv U(x, \alpha) \sim \frac{\alpha}{2\sqrt{\pi}} x^{-1/4} e^{-\frac{2}{3}\sqrt{\frac{3}{2}}x^{3/2}}$$

$$|\alpha| < 1$$

$$x \rightarrow +\infty$$

! $U(x, \alpha) \sim (-x)^{-1/4} d \cos \left(\frac{2}{3} (-x)^{3/2} - \frac{3}{4} d^2 \ln(-x) + \varphi \right)$

$$x \rightarrow -\infty$$

$$d^2 = -\frac{1}{\pi} \ln(1-\alpha^2), \quad d > 0$$

$$\varphi = -\frac{3}{2} d^2 \ln 2 - \frac{3\pi}{4} - \arg \Gamma \left(-i \frac{d^2}{2} \right) + \pi n$$

$$\text{Sign} \alpha = (-1)^n$$

Ablowitz-Segur (1977) - proof of the formula
for d

formula for φ was proven by:

Suleimanov; Clarkson, McLeod; Deift, Zhou

This type of Painlevé' appearance:

QFT - $\tilde{P_{III}}$ as reduction of sine-Gordon,
2D-Toda models

(Cecotti, Vafa, Fateev, Lukyanov...)

Differential geometry of surfaces - $\tilde{P_{III}}$
(Bobenko, et al)

Quantum cohomology, Frobenius manifolds - $\tilde{P_{IV}}, \tilde{P_V}$
(Dubrovin, et al ; Dorfmeister, Guest, Rossman)

Josephson junctions - $\tilde{P_{IV}}$ reduction of 2D elliptic
sine-Gordon (Novikov, Shagalov)

The polyelectrolytes theory (Tracy, Widom)

Hermitian Matrix Model

$$\Omega_N = \left\{ M = N \times N \text{ Hermitian matrix} \right\}$$

$$d\mu_N = \frac{1}{Z_N} e^{-N \operatorname{Tr} V(M)} DM$$

$$DM = \prod_j dM_{jj} \prod_{j < k} dM_{jk}^R dM_{jk}^I$$

$$Z_N = \int_{\Omega_N} e^{-N \operatorname{Tr} V(M)} DM - \text{partition function.}$$

$$V(\lambda) = \sum_{j=1}^{2m} t_j \lambda^j \quad t_{2m} > 0$$

Principal questions.

$$Z_N \longrightarrow ? , N \rightarrow \infty$$

eigenvalue statistics $\longrightarrow ? , N \rightarrow \infty$

Universality .

D. Bessis , C. Itzykson , J. B. Zuber

F. Dyson , M. Mehta

Quartic potential

$$V(x) = \frac{1}{4t^2} x^4 + \left(1 - \frac{2}{t}\right)x^2, \quad t > 0$$

① $Z_N \mapsto F_N = -\frac{1}{N^2} \ln Z_N$ - free energy

$$F_N(t) = F_N^{\text{Gauss}} + \int_{-\infty}^{\infty} \frac{t-s}{s^2} \left[R_N(R_{N+1} + R_{N-1}) - \frac{1}{2} \right] ds$$

$$R_n = R_n(t) :$$

$$\frac{n}{N} = R_n \left(1 - \frac{2}{t}\right) + \frac{1}{4t^2} R_n (R_{n-1} + R_n + R_{n+1})$$

$$R_n = 0 \quad n \leq 0 \quad BIZ$$

(d-PI = discrete string equ. = Freud equ.)

Also:

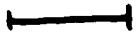
$$\{ P_n(\lambda) \}_{n=1}^{\infty} : \int_{-\infty}^{\infty} P_n(\lambda) P_k(\lambda) e^{-NV(\lambda)} d\lambda = h_n \delta_{nk}$$

$$P_n(\lambda) = \lambda^n + \dots$$

$$\lambda P_n = P_{n+1} + R_n P_{n-1}$$

$$R_n = \frac{h_n}{h_{n-1}}$$

The phase transition: $t_c = 1$, i.e.

$t > 1$ - "one cut" 

$t < 1$ - "two cut" 

$t > 1 :$

$$R_n \sim \frac{t}{3} \left(2 - t + \sqrt{(t-2)^2 + 3 \frac{n}{N}} \right)$$

$$\left(+ \sum_{k=1}^{\infty} N^{-2k} f_k \left(\frac{n}{N}; t \right) \right)$$

$t < 1 :$

$$R_n \sim t \left(2 - t + (-1)^{n+1} \sqrt{(t-2)^2 - \frac{n}{N}} \right)$$

$$N \rightarrow \infty , \quad n = N, N \pm 1$$

Bleher, I. (1998)

Scaling Theory.

$$N \rightarrow \infty, t \rightarrow 1, N^{2/3} (-1) = O(1)$$

$$R_n(x) = 1 - N^{-1/3} 2^{2/3} (-1)^n u_{HM}(x)$$

$$+ 2^{-5/3} N^{-2/3} v_{HM}(x) + O(\gamma_N)$$

$$N \rightarrow \infty, t = 1 + 2^{-2/3} N^{-2/3} x$$

u_{HM}:

$$u_{xx} = \alpha u + 2u^3 - P \underline{\underline{I}}$$

$$u(x) \sim \frac{1}{2\sqrt{\pi}} x^{-1/4} e^{-\frac{2}{3} x^{3/2}}$$

$$x \rightarrow +\infty$$

! Important: $u(x) \sim \sqrt{-\frac{x}{2}}$, $x \rightarrow -\infty$

$$(v = x + 2u^2)$$

Douglas, Seiberg, Shenker

Genkovic', Moore

Periwal, Shevitz

Bleher, I (2003)

! Also:

$$\frac{Z_N(t)}{Z_N^{\text{Gauss}}} \sim F_{\text{TW}}((t-1)2^{2/3}N^{2/3}) \\ \times Z_N^{\text{reg}}(t)$$

$$F_{\text{TW}}(x) = \exp \left\{ - \int_x^\infty (x-y) U_{\text{HH}}^2(y) dy \right\}$$

Bleher, I (2005)

②

2D Quantum Gravity

2

$$Z_N(\pm) = \int_{\{M\}} e^{-N \text{Tr} (\frac{1}{2} M^2 + \pm M^4)} dM$$

$\pm > 0$

$$\overset{\circ}{Z}_N(\pm) = \int_{\{M\}} e^{-N \text{Tr} (\frac{9}{2} M^2 + \frac{1}{4} M^4)} dM$$

$Z_N(\pm) = 2^{-\frac{N^2}{2}} \pm^{-\frac{N^2}{4}} \overset{\circ}{Z}_N\left(\frac{\pm}{2\sqrt{\epsilon}}\right)$



$$\log \frac{Z_N(t)}{Z_N(0)} \approx \sum_{g=0}^{\infty} N^{2-2g} E_g(t)$$

$$E_g(t) = \sum_{n \geq 1} (c-t)^n \frac{1}{n!} \mathcal{X}_g(n)$$

analytic : $|t| < \frac{1}{4g}$

Proven for $t > 0$ (Erdős & McLaughlin)

$\mathcal{X}_g(n) = \# \{ \text{g-maps with } n \text{ 4-valent vertices } g \}$

" = " $\# \{ \text{tilings of g-surface with } n \text{ squares } g \}$

Bessis, Itzykson, Zuber (1980)

also proven for

$$-\frac{1}{48} < t < 0 \quad (\text{Duits \& Kuijlaars})$$

$t = -\frac{1}{48}$ - singularity of $E_g(t)$.

Double scaling near $t = -\frac{1}{48}$.

$$t = -\frac{1}{48} - 2^{-19/5} N^{-4/5} \propto$$

$$R_N \sim 2 - c N^{-3/5} u(x), \quad c = 2^{3/5} 3^{3/5}$$

$$u_{xx} = 6u^2 + x - PI$$

$$u(x) \sim \sqrt{-\frac{x}{6}} + \sum_{p=1}^{\infty} (-x)^{\frac{1}{2} - \frac{5}{2} p} c_p, \quad x \rightarrow \infty$$

$$-\frac{\pi}{5} < \arg z < \frac{4\pi}{5}$$

Brézin \& Kazakov; Douglas \& Shenker;
 Gross \& Migdal; David; Kitaev; Fakas, Kitaev, I;
 Duits \& Kuijlaars

Power



$$u(x) = e^{i \frac{\pi}{3}} \sqrt{\frac{1+x}{6}}$$

$$+ \frac{1}{\sqrt{8\pi}} e^{-i \frac{\pi}{2} \theta_0} \left(\frac{2}{3}\right)^{1/8} i |x|^{-1/8} \\ \times \exp \left\{ - \frac{8i}{5} \left(\frac{3}{2}\right)^{1/4} |x|^{5/4} \right\}$$

$$x \rightarrow \infty, \arg x = \frac{7\pi}{5}$$

(1 + o(1))

A. Kapur.

(triply truncated solution)

$$\frac{1}{N^2} \log Z_N(\pm) \xrightarrow[N \rightarrow \infty]{\Downarrow} -F(x)$$

$$\text{, } \pm = -\frac{1}{48} - 2^{-1/5} N^{-4/5} x$$

$$F''(x) = -U(x)$$

Matching with the non-critical expansion.

$$x_g(n) \sim \frac{48^n n!}{n^{1+\frac{5}{2}(1-g)}} c_g \iff E_g(\pm) \sim \alpha_g \left(\pm + \frac{1}{48}\right)^{\frac{5}{2}(1-g)}$$

(Di Francesco , Gao)



$$\sum_{g=0}^{\infty} N^{2-2g} E_g(\pm) \sim \sum_{g=0}^{\infty} N^{2-2g} \alpha_g \left(\pm + \frac{1}{48}\right)^{\frac{5}{2}(1-g)}$$

$$= \sum_{g=0}^{\infty} c_g(-\infty)^{\frac{5}{2}-\frac{5}{2}g}$$

↓

$$\int (x-y) u(y) dy$$

= non-perturbative definition

of the partition function

of 2D quantum gravity.

More on this type of the Painlevé' equations appearance.

1. Other graph counting problems. Intersection theory of the moduli spaces of curves.
Witten - Kontsevich theorem.

Di Francesco, Ginsparg , J. Zinn-Justin

Itzykson , Zuber , Eguchi , Kamada , S-K Yang

Olkunov , Pandharipande ,

Adler , Van Moerbeke ,

Gao , Garoufalidis , Marino , Brèzen , Hikami

2. Hele - Shaw flow , normal matrix model ,
conformal maps & integrable hierarchies
Zabrodin , Wiegmann , Miheev - Weinstein and others.

Painlevé before large N limit

$$R_n(\pm, N) = \frac{\pm}{\sqrt{N}} u(x), \quad x = (\pm - 2)N^{1/2}$$

$$u(x) = u(x, n) :$$

$$\begin{aligned} \frac{d^2 u}{dx^2} = & \frac{1}{2u} \left(\frac{du}{dx} \right)^2 + \frac{3}{2} u^3 + 4xu^2 \\ & + 2(x^2 + \frac{n}{2})u - \frac{n^2}{2u} \end{aligned}$$

PIV

Kitaev (1991)

More on this:

Magnus, Chen, Ismail, Fobas, Kitaev, I

- Painlevé' for semiclassical
orthogonal polynomials

Adler, Van Moerbeke - matrix integrals and
Painlevé'

Forrester, Frankel, Witte - Σ -function theory
approach

Kanzieper - replica

3. Eigenvalue statistics.

Bulk:

$$E_0^{(N)} \left(\lambda_0 - \frac{c\alpha}{NG(\lambda_0)}, \lambda_0 + \frac{c\alpha}{NG(\lambda_0)} \right)$$

$$\mapsto E_0(x)$$

$$N \rightarrow \infty$$

$$E_0(x) = \det \left(1 - K_{\sin} \Big|_{(0, 2x)} \right)$$

$$= \exp \left(\sum_0^{\infty} \frac{\beta(t)}{t} dt \right)$$

$$(xz'')^2 + 4(4z - xz' - (z')^2)(z - xz') = 0$$

$$- P \bar{V}$$

$$\beta(x) \sim -\frac{2}{\pi}x, x \rightarrow 0.$$

JMMS

$$E_0(x) \sim x^{-1/4} e^{-\frac{1}{2}x^2 + c_0}, \quad x \rightarrow +\infty$$

$$c_0 = 3 \sum'_{n=1} (-1)^n + \frac{1}{12} \ln 2$$

Dyson, Widom, Suleimanov, Krasovsky, Ehrhardt

Edge:

$$g(\lambda) \sim c(\lambda - \ell)^{\frac{4k+1}{2}} \quad \xrightarrow{\ell}$$

$$\text{Prob} \left(c N^{\frac{2}{4k+3}} (\lambda_{\max}^{(N)} - \ell) < \alpha \right)$$

$$\xrightarrow[N \rightarrow \infty]{} F_{TW}^{(k)}(x)$$

$$F_{TW}^{(0)}(x) = f_{TW}(x) \quad (\text{. . . , } \cdot \cdot \cdot)$$

$$F_{TW}^{(k)}(x) = \text{in terms of } P_{\parallel}^{(k)}$$

Claeys, Krasovsky, I

More on this type of Painleve'
appearance

Tracy, Widom

Adler, Van Moerbeke

Forrester, Witte

Osipov, Kan zieper

An explanation of the appearance of Painlevé' equations

1. A Riemann-Hilbert representation of the original physical problem
2. Direct identification of the original RH problem with the RH problem from the Painlevé' list - Painlevé' equations before or after the large N limit.
3. Appearance of the Painlevé' RH problem as a parametrix of the asymptotic solution of the original RH problem - Double scaling limits, transition asymptotic regimes
 - ≡ appearance of the Airy, Bessel, etc. in the asymptotic analysis of oscillatory integrals.

1.1 The RH representation of the correlation functions.

- Determinant representation of the correlation functions:

$$\boxed{\det(1 - K)} \quad K: L_2(\Gamma) \rightarrow L_2(\Gamma)$$

$$K(x, x') = \frac{f^T(x) h(x')}{x - x'}$$

$$f(x) = \begin{pmatrix} f_1 \\ \vdots \\ f_m \end{pmatrix}, \quad h(x) = \begin{pmatrix} h_1 \\ \vdots \\ h_m \end{pmatrix}$$

(integrable integral operators)

Examples

Sine-Kernel, Airy-Kernel, Bessel-Kernel,
 Hermite-Kernel, Darboux-Kernel

Painlevé-kernels

Gaudin, Lenard,

Jimbo, Miwa, Mori, Sato

Tracy, Widom

Forrester

Izergin, Korepin, Slavnov, I

Akemann; Bowick, Brézin, Marinari, Parisi,

Bleher, I

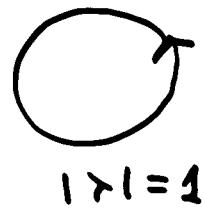
Claeys, Kuipers, Vanlessen

Claeys, Krasovsky, I

Kuipers, Östensson, I

! Toeplitz determinants

$$\varphi_j = \int_{|\gamma|=1} \varphi(\gamma) \gamma^{-j-1} \frac{d\gamma}{2\pi i}$$



$$T_n[\varphi] = \{\varphi_{j-k}\}_{j,k=0,\dots,n-1}$$

$$\det T_n[\varphi] = \det (1 - K_n)$$

$$K_n : L_2(\Gamma) \rightarrow \Gamma : |\gamma|=1$$

$$K_n(\gamma, \gamma') = \frac{(\gamma/\gamma')^n - 1}{\gamma - \gamma'} \cdot \frac{1 - \varphi(\gamma')}{2\pi i}$$

(Defit.)

\Rightarrow correlation functions of Ising and quantum spin model, random permutations, random tiling

• The RH representation:

$$R = (1 - k)^{-1} k, \quad R(\lambda, \lambda') = \frac{F^T(\lambda) H(\lambda')}{\lambda - \lambda'}$$

$$F(\lambda) = Y_+(\lambda) f(\lambda), \quad H(\lambda) = (Y_+^T(\lambda))^{-1} h(\lambda)$$

$$\cdot Y(\lambda) \in H(C \setminus \Gamma)$$

$$\cdot Y_-(\lambda) = Y_+(\lambda) G_e(\lambda), \quad \lambda \in \Gamma$$

$$\cdot Y(\lambda) \sim I, \quad \lambda \rightarrow \infty$$

$$G_e(\lambda) = I_m + 2\pi i f(\lambda) h^T(\lambda)$$

$$(G_{e,j_k} = \delta_{j_k} + 2\pi i f_j(\lambda) h_k(\lambda))$$

$\therefore T_{j_1, j_2, \dots, j_n} = (f_{j_1}, \dots, f_{j_n}) h_{j_1} \dots h_{j_n}$

Remark. An alternative RH problem for Toeplitz determinants.

$$G_T(\gamma) = \begin{pmatrix} 2-\varphi & \gamma^n(\varphi-1) \\ \gamma^{-n}(1-\varphi) & \varphi \end{pmatrix}$$

$$= \begin{pmatrix} \gamma^n & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\gamma^{-n}\varphi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma^{-n} & 0 \\ -1 & \gamma^n \end{pmatrix}$$

$$\tilde{Y}(\gamma) := \begin{cases} Y(\gamma) \begin{pmatrix} \gamma^n & -1 \\ 1 & 0 \end{pmatrix} & |\gamma| < 1 \\ Y(\gamma) \begin{pmatrix} \gamma^n & 0 \\ 1 & \gamma^{-n} \end{pmatrix} & |\gamma| > 1 \end{cases}$$

$$\cdot \tilde{Y}(\lambda) \in H(C \setminus \Gamma) \quad \Gamma : |\lambda| = 1$$

$$\cdot \tilde{Y}_+(\lambda) = \tilde{Y}_-(\lambda) \begin{pmatrix} 1 & \lambda^{-n} \varphi(\lambda) \\ 0 & 1 \end{pmatrix} \quad \lambda \in \Gamma$$

$$\cdot \tilde{Y}(\lambda) = (I + O(\frac{1}{\lambda})) \lambda^{n \mathfrak{Z}_3}, \lambda \rightarrow \infty$$

$$\mathfrak{Z}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Important:

$$\tilde{Y}_{12}(\lambda) = P_n(\lambda) - OPUC:$$

$$\int_{\Gamma} P_n(\lambda) \overline{P_j(\lambda)} \varphi(\lambda) \frac{d\lambda}{i\lambda} = h_n \delta_{nj}, \quad P_n(\lambda) = \lambda^n + \dots$$

$$\text{also: } \det T_{n+1} / \det T_n \equiv \frac{1}{2\pi} h_n = \tilde{Y}_{12}(0)$$

Baik, Deift, Johansson (1999)

1.2 The RH representation of orthogonal polynomials (\Rightarrow Hermitian matrix model)

$$\Gamma = \mathbb{R}, \quad G_\ell(\lambda) = \begin{pmatrix} 1 & \omega(\lambda) \\ 0 & 1 \end{pmatrix}$$

$$\omega(\lambda) = e^{-V(\lambda)}, \quad V(\lambda) = \sum_{j=1}^{2m} t_j \lambda^j, \quad t_{2m} > 0$$

- $Y(\lambda) \in H \subset \mathbb{C} \setminus \mathbb{R}$

- $Y_+(\lambda) = Y_-(\lambda) G_\ell(\lambda)$

- $Y(\lambda) = (I + O(\lambda)) \lambda^{n \delta_3}, \quad \lambda \rightarrow \infty$

$$\delta_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Y_{12}(\gamma) = P_n(\gamma) - \text{OPRL:}$$

$$\int_{-\infty}^{\infty} P_n(\gamma) P_{\ell}(\gamma) \omega(\gamma) d\gamma = h_n \delta_{n\ell}, \quad P_n(\gamma) = \gamma^n + \dots$$

also:

$$D_{n+1}/D_n = h_n = \frac{i}{2\pi} \lim_{\lambda \rightarrow \infty} \lambda^{n+1} Y_{12}(\gamma)$$

Note:

$$Z_N = \text{const} \int \dots \int \prod_{j < k} (\gamma_j - \gamma_k)^2 \prod_{j=1}^N \omega(\gamma_j) d\gamma_1 \dots d\gamma_N$$

$$= \text{const } N! D_N$$

$$D_n = \det \left\{ \int_{-\infty}^{\infty} \gamma^{k+j} \omega(\gamma) d\gamma \right\}_{k,j=0 \dots n-1}$$

- Hankel determinant.

Fabas, Kitau, I
(1991)

No explanation !

(Original problem - not integrable)

- Three-dimensional wave collapse and P_{II} .
(Zakharov, Kuznetsov, Musher; Novokshenov)
- Soshnikov's result
- Current fluctuations in ASEP
(Tracy, Widom)



The Painlevé transcendents are indeed
the nonlinear special functions.

Appendix 1.

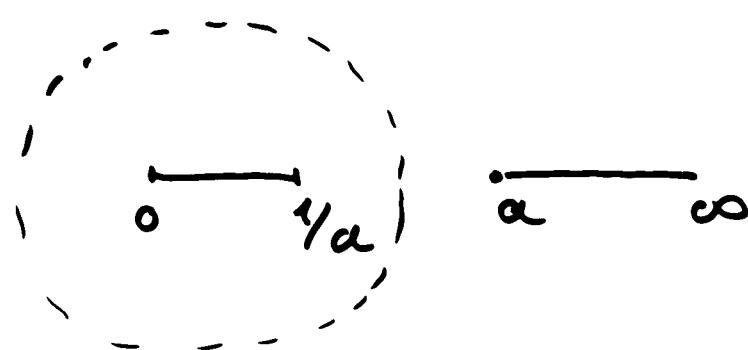
Ising model. Painlevé VI.

(Jimbo - Miwa R1)

$$\langle \delta_{00} \delta_{NN} \rangle = \det T_N[\varphi]$$

$$\varphi(\lambda) = \left(\frac{\alpha - \lambda^{-1}}{\alpha - \lambda} \right)^{1/2} \quad \alpha = \sinh \frac{2E^v}{kT} \sinh \frac{2E^b}{kT}$$

$$T > T_c \Rightarrow \alpha > 1$$



Wu, McCoy (1966-68)

Observe :

$$G \tilde{e} \tilde{Y}(\gamma) = \begin{pmatrix} 1 & \gamma^{-N}\varphi \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & \gamma^N\varphi^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \gamma^N\varphi \end{pmatrix}$$

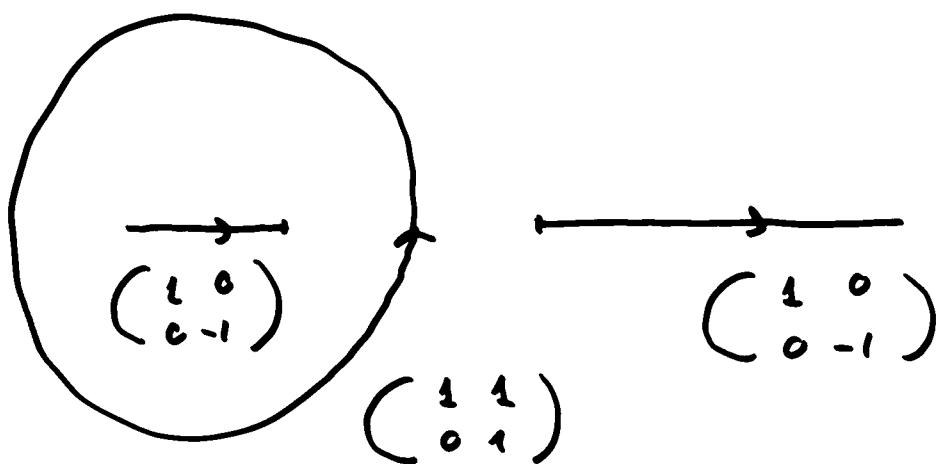
Put

$$\Psi(\gamma) := \tilde{Y}(\gamma) \Psi_0(\gamma)$$

$$\Psi_0(\gamma) = \begin{pmatrix} 1 & 0 \\ 0 & \gamma^N\varphi^{-1} \end{pmatrix}$$



$$\bullet \quad \Psi_+(\lambda) = \Psi_-(\lambda) G_\Psi \quad \lambda \in \Gamma_\Psi$$



in particular, $G_\Psi \equiv \text{const}$

Observe also,

$$\frac{d\Psi_0}{d\lambda} = \left(\frac{A_1^0}{\lambda} + \frac{A_2^0}{\lambda-a} + \frac{A_3^0}{\lambda-a^{-1}} \right) \Psi_0$$

$$A_1^0 = \left(N + \frac{1}{2} \right) \begin{pmatrix} 0 & 0 \\ c_1 & \end{pmatrix}, \quad A_{2,3}^0 = \pm \frac{1}{2} \begin{pmatrix} 0 & 0 \\ c_1 & \end{pmatrix}$$

↓ !

$$\frac{d\Psi}{d\lambda} = \left(\frac{A_1}{\lambda} + \frac{A_2}{\lambda - \alpha} + \frac{A_3}{\lambda - \alpha^{-1}} \right) \Psi$$

$$\frac{d\Psi}{d\alpha} = \left(\frac{B_1}{\lambda - \alpha} + \frac{B_2}{\lambda - \alpha^{-1}} \right) \Psi$$

↓

$\det T_N(z)$ = ζ -function of the
sixth Painlevé' equation

$$z = \alpha^{-2} = \left(\sinh \frac{2E^v}{kT} \sinh \frac{2E^b}{kT} \right)^{-2}.$$

! Remark

$$\left\{ \begin{array}{l} \Psi_\lambda = A(\lambda) \Psi \\ \Psi_a = B(\lambda) \Psi \\ \underline{\Psi_{N+1} = C(\lambda) \Psi} \end{array} \right.$$

$$A_a - B_\lambda = [B, A] - P \bar{V}$$

$$C_\lambda + C A_N - A_{N+1} C - d - P \bar{V}$$

$$C_a + C B_N - B_{N+c} C - T \text{oda}$$

Appendix 2

Ising model. Double scaling
and Painlevé III

(Tracy - McCoy - Witten $\Rightarrow \tau^+$)

$$\tilde{X}(z) := \begin{cases} Y(z) \begin{pmatrix} 1 & z^n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_+^{-1} & 0 \\ 0 & u_+ \end{pmatrix} & |z| < 1 \\ Y(z) \begin{pmatrix} 1 & 0 \\ z^{-n} & 1 \end{pmatrix} \begin{pmatrix} u_- & 0 \\ 0 & u_-^{-1} \end{pmatrix} & |z| > 1 \end{cases}$$

$$\Psi(z) = u_+ u_-$$

$$\tilde{X}_-(z) = \tilde{X}_+(z) \begin{pmatrix} 0 & -\frac{u_+}{u_-} z^n \\ \frac{u_-}{u_+} z^{-n} & 1 \end{pmatrix}$$

$$= \tilde{X}_+(z) \begin{pmatrix} 1 & -\frac{u_+}{u_-} z^n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{u_-}{u_+} z^{-n} & 1 \end{pmatrix}$$

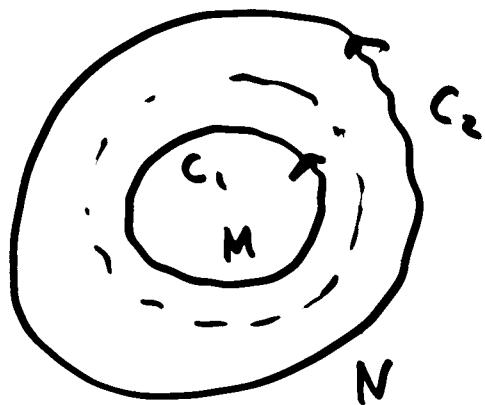
2.

$$\tilde{X}(z) \rightarrow X(z) = \begin{cases} \tilde{X} M & |z| < 1 \\ \tilde{X} N^{-1} & |z| > 1 \end{cases}$$

$$M = \begin{pmatrix} 1 & -\frac{u_+}{u_-} z^n \\ c & 1 \end{pmatrix}$$

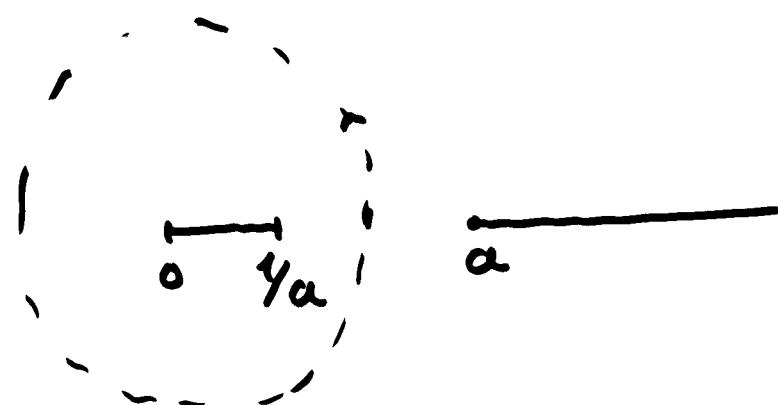
$$N = \begin{pmatrix} 1 & 0 \\ \frac{u_-}{u_+} z^{-n} & 1 \end{pmatrix}$$

$$X_- = X_+ G_+$$



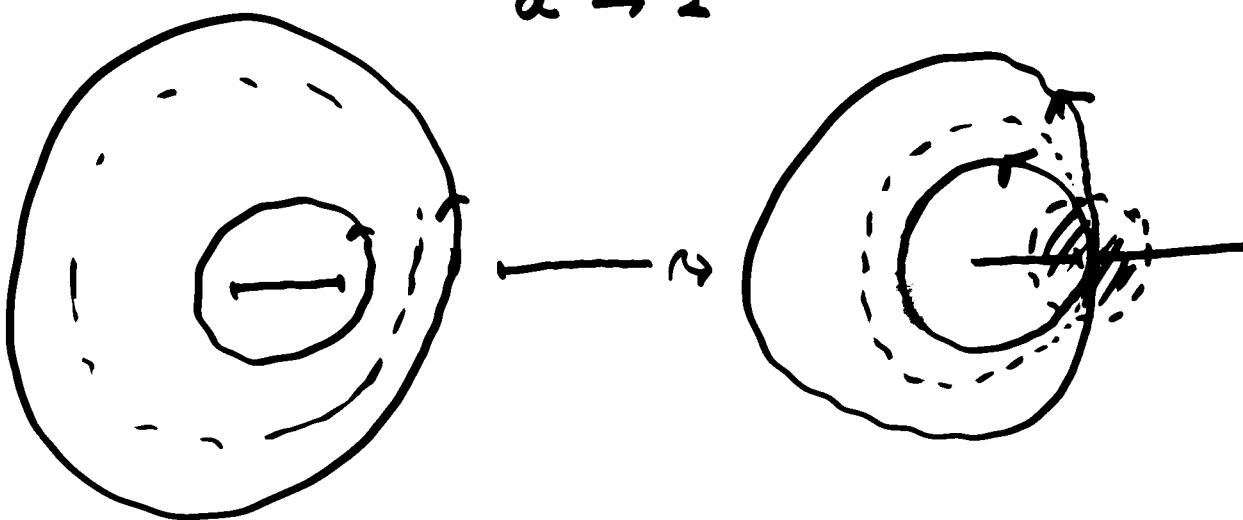
Isting: $\varphi(z) = \left(\frac{a - z^{-1}}{a - z} \right)^{\frac{1}{2}}$

$$= \left(\frac{a}{z} \right)^{\frac{1}{2}} \left(\frac{z - a^{-1}}{a - z} \right)^{\frac{1}{2}} \quad a > 1$$



$$u_+ = \left(\frac{a}{a-z} \right)^{\frac{1}{2}} \quad u_- = \left(\frac{z-a^{-1}}{z-a} \right)^{\frac{1}{2}}$$

$$a \rightarrow 1$$



$$a = 1 + \frac{x}{n}, \quad z = 1 + \frac{i\xi}{n}$$

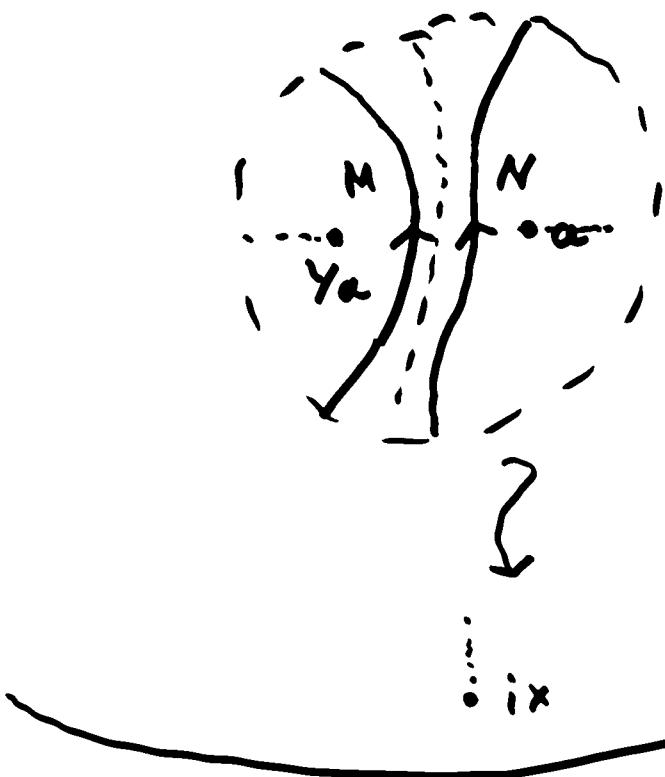
$$z^n \sim e^{i n \xi}$$

$$u_+ \sim \frac{n^{1/2}}{(x-i\xi)^{1/2}}$$

$$u_- \sim \frac{(i\xi + x)^{1/2}}{n^{1/2}}$$

$$M \sim \begin{pmatrix} 1 & -\frac{e^{i n \xi}}{(x^2 + \xi^2)^{1/2}} \\ 0 & 1 \end{pmatrix}$$

$$N \sim \begin{pmatrix} 1 & 0 \\ \frac{e^{-i n \xi}}{(x^2 + \xi^2)^{-1/2}} & 1 \end{pmatrix}$$



$$\left(\begin{array}{cc} 1 & \frac{e^{i\xi}}{(x^2 + \xi^2)^{1/2}} \\ 0 & 0 \end{array} \right)$$

$$\left(\begin{array}{cc} 1 & 0 \\ e^{-i\xi} (x^2 + \xi^2)^{1/2} & 1 \end{array} \right)$$

$$G_c(\xi) = e^{d(\xi)\beta_3} G_0 e^{-d(\xi)\beta_3}$$

$$d(\xi) = \frac{i\xi}{2} - \frac{1}{4} \ln(x^2 + \xi^2)$$

$\boxed{\frac{d}{d\xi} d(\xi) = C_1 + \frac{C_2}{\xi - ix} + \frac{C_3}{\xi + ix} \Rightarrow \bar{PV} \approx \bar{P\|I\|}}$

$$\overset{\circ}{\psi}(\xi) := \overset{\circ}{X}(\xi) e^{d(\xi) \beta_3} :$$

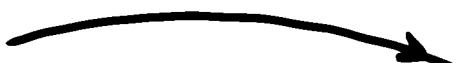
$$\overset{\circ}{\psi}_- - \overset{\circ}{\psi}_+ G_0$$

(11)



$$\cdot \overset{\circ}{\psi}(\xi) = (I + \sum_{n=1}^{\infty} \dots) \\ \times e^{d(\xi) \beta_3}$$

(10)



$$\frac{\partial \overset{\circ}{\psi}}{\partial \zeta} \overset{\circ}{\psi}^{-1} = \frac{i}{2} \beta_3 + \frac{A_1}{\zeta - ix} + \frac{A_2}{\zeta + ix} \equiv A(\zeta; m_2)$$

$$\frac{\partial \overset{\circ}{\psi}}{\partial x} \overset{\circ}{\psi}^{-1} = - \frac{i A_1}{\zeta - ix} + \frac{i A_2}{\zeta + ix} \equiv U(\zeta; m_1)$$

↓

$$A_x - U_\zeta = [U, A] \Leftrightarrow P \bar{U} \quad \text{Sei}$$

$$u(x) := (m_2)_{12} \\ \mathcal{J}. M.$$