Supersymmetry for disordered systems with interaction

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Outline

- Introduction
  - Typical phenomena in disordered electron systems
  - Diffusion modes and nonlinear sigma-model
  - Normalization-different techniques
- Supersymmetric Model
  - Non-interacting part
  - Interaction part
  - Sigma-model with interaction
  - Testing the model
- Conclusion

The system

Interacting fermions in a random disorder potential at low temperatures

- Weak delta-correlated impurity potential

$$\langle U(\mathbf{r})U(\mathbf{r}')\rangle = \frac{1}{2\pi\nu\tau}\delta(\mathbf{r}-\mathbf{r}') \text{ and } \varepsilon_F\tau \gg 1$$

- Low temperatures:

$$T\tau \ll 1$$

Aim: Calculation of disorder averaged correlation functions

Typical Phenomena

Weak localization:

$$\sigma \overset{d=2}{=} \sigma_0 - \frac{e^2}{\pi^2} \ln\left(\frac{L}{l}\right), \qquad L^d : \text{system size}$$

Interaction corrections:

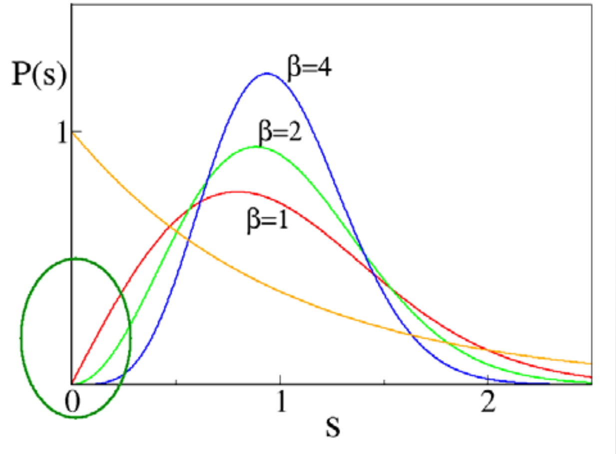
$$\sigma \overset{d=2}{=} \sigma_0 - \frac{e^2}{2\pi^2} \ln\left(\frac{1}{T\tau}\right), \qquad (T\tau \ll 1)$$

Wigner-Dyson statistics:

Level Distribution,

P(s): distance to nearest neighbor

$$P(s) \propto s^\beta \qquad \text{for } s \ll 1, \quad \beta = 1, 2 \text{ or } 4$$



Localization, Integer Quantum Hall effect, Metal-Insulator transition, Superconductor-Insulator transition …

Theoretical Description

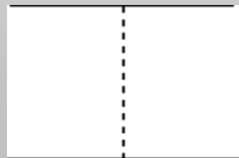
- Partition function for a specific disorder configuration:

$$
\begin{aligned}
Z &= \int D(\chi\chi^*)\ \exp\left(-S[\chi,\chi^*]\right), \\
S &= S_{free} + S_{dis} + S_{int} \\
S_{dis} &= \int_0^\beta d\tau \int d\mathbf{r}\, \chi^*_\alpha(\mathbf{r},\tau) U(\mathbf{r}) \chi_\alpha(\mathbf{r},\tau) \\
S_{int} &= \frac{1}{2}\int_0^\beta d\tau \int d\mathbf{r} d\mathbf{r}'\, \chi^*_\alpha(\mathbf{r},\tau)\chi^*_\beta(\mathbf{r}',\tau) V(\mathbf{r}-\mathbf{r}')\chi_\beta(\mathbf{r}',\tau)\chi_\alpha(\mathbf{r},\tau)
\end{aligned}
$$

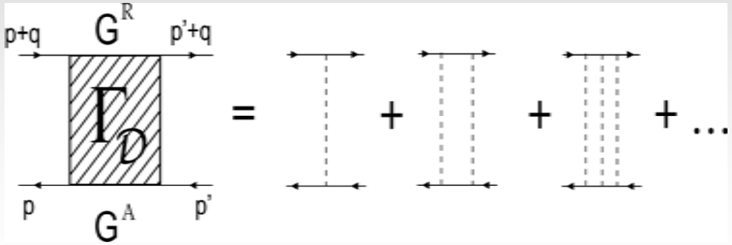
χ: Grassmann fields $(\chi_i\chi_j = -\chi_j\chi_i)$ with $\chi(\mathbf{r},\tau) = -\chi(\mathbf{r},\tau+\beta)$, α,β: Spin

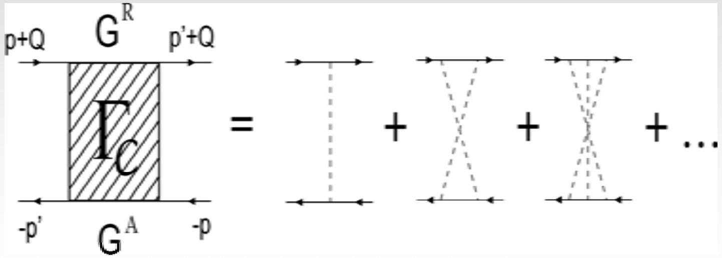
- Impurity averaging (schematic):

$$
\left\langle \exp\left[-\int \chi^*(\mathbf{r},\tau) U(\mathbf{r}) \chi(\mathbf{r},\tau)\right]\right\rangle_{dis} = \exp\left[\frac{1}{4\pi\nu\tau}\int \chi^*(\mathbf{r},\tau)\chi(\mathbf{r},\tau)\chi^*(\mathbf{r},\tau')\chi(\mathbf{r},\tau')\right]
$$

Averaging generates "interaction": 

- Diffusion modes (Diffusons, Cooperons) determine low-T behavior



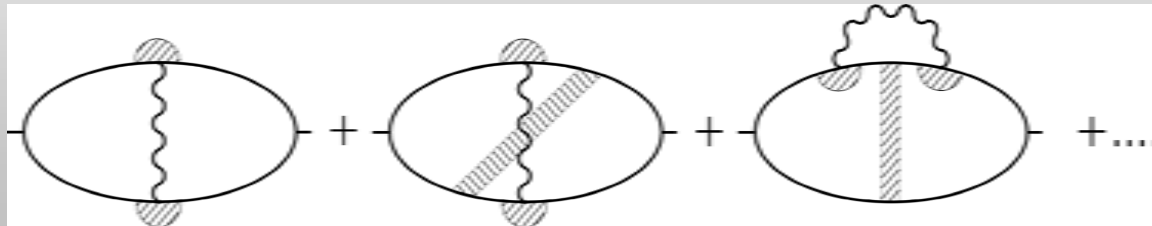


$$\mathcal{C}(Q=(\mathbf{Q},\omega)) = \frac{1}{D\mathbf{Q}^2 - i\omega}, \qquad \mathcal{D}(q=(\mathbf{q},\omega)) = \frac{1}{D\mathbf{q}^2 - i\omega}$$

Weak localization:

$$\delta\sigma = -\frac{2e^2 D}{\pi}\int (d\mathbf{Q})\mathcal{C}(\mathbf{Q},\omega)$$

- Diagrammatics:
  - Example: Interaction (Altshuler-Aronov) corrections to conductivity



- Diagrammatic perturbation theory is powerful, but
  - higher orders in perturbation theory are very cumbersome,
  - perturbation theory does not lead to, e.g., Wigner-Dyson statistics.
- Different approach:
  - Effective low energy theory ->non-linear sigma-models
  - Hubbard-Stratonovich transformation with matrix field Q
  - Integrating out fast electron degrees of freedom
  - Saddle point manifold
- Final form (non-interacting, supersymmetric)

$$\boxed{Z = \int_{Q^2=1} \mathcal{D}Q \; \exp[-F] \qquad F = \frac{\pi\nu}{4}\int_r \mathrm{Str}\;[D(\nabla Q)^2 + 2i\omega\Lambda Q]}$$

  - Renormalization group analysis
  - Calculations beyond perturbation theory

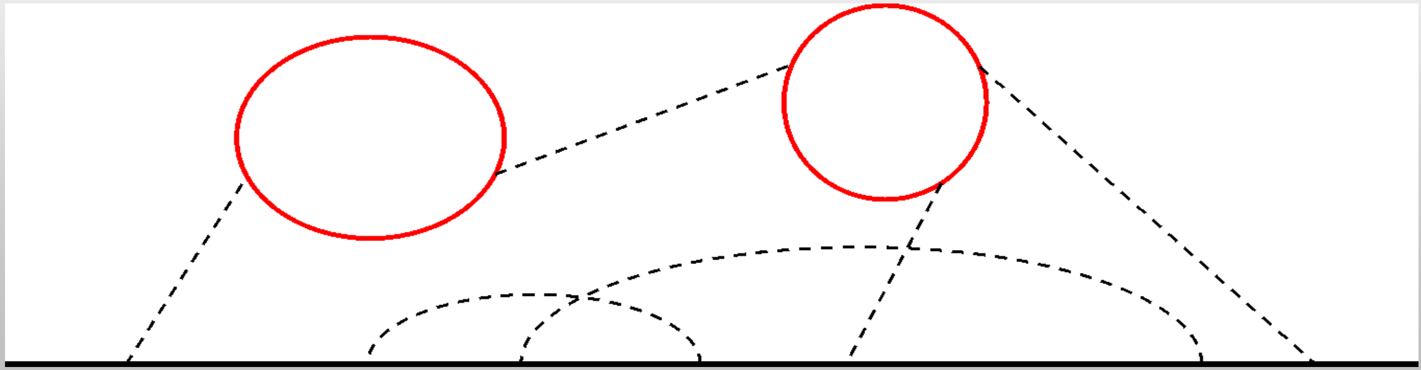
Normalization

- Disorder averaged correlation functions: Example-Green's function

$$\langle G \rangle_{dis} \propto \left\langle \frac{1}{Z[U]} \int d\chi d\chi^* \; \chi\chi^* \; \exp[-S[\chi,\chi^*,U]] \right\rangle_{dis}$$

  - Normalization depends on disorder configuration.
  - Averaging does not correspond to a simple Gaussian integral

- Diagrammatically
  - Part of $\langle \int d\chi d\chi^* \; \chi\chi^* \; \exp[-S[\chi,\chi^*,U]] \rangle_{dis}$:



Different techniques I

- Replica trick: Use relation $\ln Z = \lim_{N\to 0} (Z^N - 1)/N$, first compute

$$\langle Z^N \rangle_{dis} = \left\langle \int \prod_{i=1}^{N} D(\chi^i, \chi^{i*}) \mathrm{e}^{-\sum_{i=1}^{N} S[\chi^{i*}, \chi^i, U]} \right\rangle$$

Edwards, Anderson 75

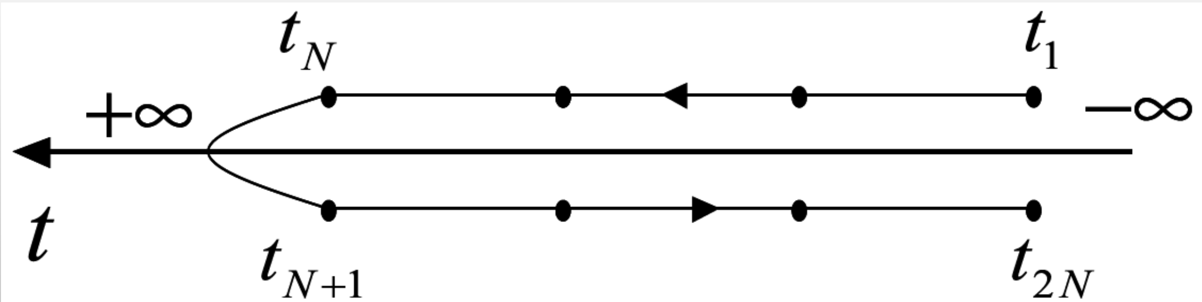
Wegner 79-

Efetov, Larkin, Khmelnitskii 80

Finkel'stein 84

  - Correlation functions via $\partial_J \ln Z = \partial_J Z/Z$. Vacuum loops carry factor N.

- Real-time (Keldysh) approach: Special time contour.



Keldysh 65

Horbach Schön 92

Kamenev Andreev 99

Chamon, Ludwig, Nayak 99

Different techniques II

- Replica Sigma-model with interaction (Finkel'stein 84)

$$Z = \int DQ \, e^{-F[Q]}$$

$$F = \frac{\pi\nu}{4} \int d\mathbf{r} \left\{ Tr[D(\nabla Q)^2 - 4EQ] + \Gamma_2 Q\gamma_2 Q - \Gamma_1 Q\gamma_1 Q \right\}$$

$$Q\gamma_1 Q = 2\pi T \sum_{abcd,i,\alpha\beta} Q_{ab}^{ii,\alpha\alpha} \delta_{a+c,b+d} Q_{cd}^{ii,\beta\beta} \qquad Q\gamma_2 Q = 2\pi T \sum_{abcd,i,\alpha\beta} Q_{ab}^{ii,\alpha\beta} \delta_{a+c,b+d} Q_{cd}^{ii,\beta\alpha}$$

$$Q = U\Lambda U^\dagger \qquad UU^\dagger = 1 \qquad \Lambda_{nk} = \operatorname{sgn} \omega_n \delta_{nk}$$

- Renormalizable model, coupled RG equations for Γ_1, Γ_2, $(\nu D)^{-1}$, z (first order in $(\nu D)^{-1}$, all orders in Γ_1, Γ_2)

Different techniques III

- Main idea of supersymmetry (non-interacting systems):
  Rewrite $Z[U]^{-1}$ as a Gaussian integral over bosonic variables

$$I_{boson} = \frac{1}{\pi}\int dz dz^* \, \mathrm{e}^{-az^*z} = \frac{1}{a}\,, \qquad I_{fermion} = \int d\chi^* d\chi \, \mathrm{e}^{-a\chi^*\chi} = a$$

$$\boxed{Z^{-1} \simeq (I_{fermion})^{-1} = I_{boson}}$$

Efetov 83-

Weidenmüller, Verbaarschot, Zirnbauer 85

- Resulting theory contains commuting ($zz^*=z^*z$) and anticommuting ($\chi\chi^*=-\chi^*\chi$) variables .
  - Fermionic and bosonic vacuum loops cancel.

- **Strength of supersymmetry:** non-perturbative analysis
  (examples: Level-Statistics, localization)

Overview

- **Three Variants:** Replica, Keldysh, Supersymmetry

	perturbative	non-perturbative
no interaction	Replica, SUSY, Keldysh	SUSY (Replica*,Keldysh**)
interaction	Replica, Keldysh	?

- **Different strategies to fill ?:**
    1) Find a consistent description using Replica, Keldysh approaches for non-interacting systems first. If successful, inclusion of interactions poses (formally) no problem.

       [Verbaarschot et al. (1985*),Kamenev et al. (1999*), Yurkevich et al. (1999*), Zirnbauer (1999*), Altland et al. (2000**,2005*), Kanzieper (2002*)]
    2) Find a way how to include interactions into SUSY scheme. [GS, Efetov (2005)]
    3) New ideas

Supersymmetric model with interaction

Abstract

- We construct a supersymmetric sigma-model for weakly disordered systems.
- The model includes a weak short-range interaction.
- Due to supersymmetry disorder averaging poses no problem.
- We test the model by perturbation theory and renormalization group calculations.
- The model is intended as a tool for non-perturbative calculations.

GS, Efetov, Phys. Rev. B 71, 134203 (2005)

Derivation: Non-interacting part

- Partition function:

$$Z = \int D(\chi\chi^*)\, \mathrm{e}^{-S_0[\chi,\chi^*,U]}$$

$$S_0 = -i \int d\mathbf{r}\, \bar{\chi}_\alpha(\mathbf{r}) \left[iE + \hat{\xi}_p + U(\mathbf{r})\right] \chi_\alpha(\mathbf{r}),$$

$$\chi_n^* \to i\chi_n^* \mathrm{sgn}\,(\omega_n) = -i\bar{\chi}_n, \quad \chi(\mathbf{r}) = \begin{pmatrix} \vdots \\ \chi(\mathbf{r},-1) \\ \chi(\mathbf{r},0) \\ \chi(\mathbf{r},1) \\ \vdots \end{pmatrix}, \quad \hat{\xi}_p = \frac{\hat{p}^2}{2m} - \mu, \quad E_{nk} = -\omega_n \delta_{nk}$$

- Supersymmetric model:

$$\chi(\mathbf{r}) \to \psi(\mathbf{r}), \qquad \psi(\mathbf{r},n) = \begin{pmatrix} \chi(\mathbf{r},n) \\ S(\mathbf{r},n) \end{pmatrix},$$

$$S_0'[\psi,\bar{\psi},U] = S_0[\chi,\chi^*,U] + S_0[S,S^*,U] = -i \int d\mathbf{r}\, \bar{\psi}_\alpha(\mathbf{r}) \left[iE + \hat{\xi}_p + U(\mathbf{r})\right] \psi_\alpha(\mathbf{r})$$

$$Z' = \int D(\psi,\bar{\psi})\, \mathrm{e}^{-S_0'[\psi,\bar{\psi},U]} = 1$$

- Disorder average and HS transformation with supermatrix Q:

$$\left\langle \exp\left[-\int \bar{\psi}(\mathbf{r})U(r)\psi(\mathbf{r})\right]\right\rangle_{dis} = \exp\left[\frac{1}{4\pi\nu\tau}\int \left(\bar{\psi}(\mathbf{r})\psi(\mathbf{r})\right)^2\right]$$

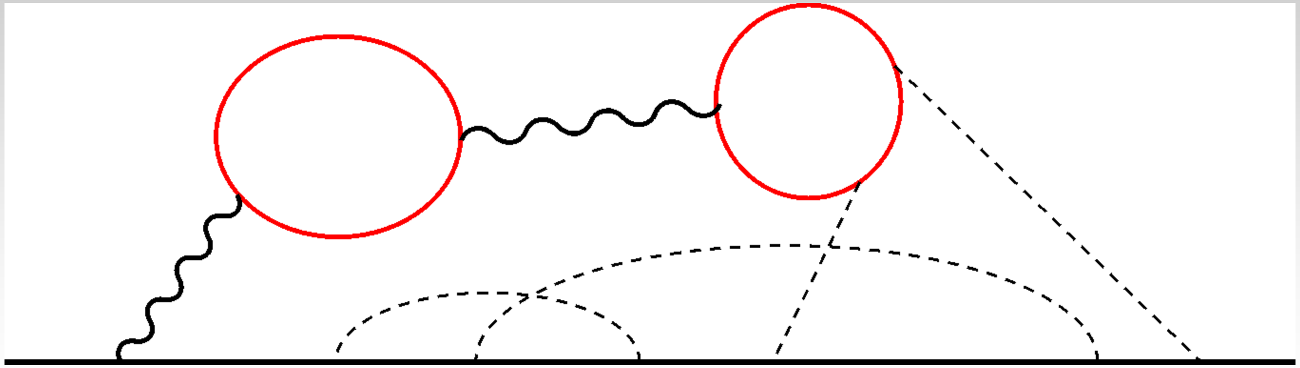
$$= \int DQ \, \exp\left[\frac{i}{2\tau}\int \bar{\psi}(\mathbf{r})Q(\mathbf{r})\psi(\mathbf{r}) - \frac{\pi\nu}{4\tau}\int \mathrm{Str}\,\left(Q(\mathbf{r})Q(\mathbf{r})\right)\right]$$

- Low energy theory is non-linear sigma-model:

$$Z' = \int DQ \, \mathrm{e}^{-F}$$

$$F = \frac{\pi\nu}{4}\int d\mathbf{r} \, \mathrm{Str}\left[D(\nabla Q(\mathbf{r}))^2 - 4EQ\right].$$

Supersymmetry and interactions: Where is the problem?



- Diagrammatically: Supersymmetry (used naively) takes away all loops!
- The interacting theory is a many-body theory-single particle picture is not sufficient.
- There is no canonical way to make the interacting theory supersymmetric.

Instead of deriving a model, we introduce an artificial supersymmetric model.
We have to guess a proper interaction term.

Interaction part

- Original interaction term:

$$S_{int} = -\frac{T}{2}\int \bar{\chi}^{\alpha}(\mathbf{r})\Delta^{j}\chi^{\alpha}(\mathbf{r})\; V_0(\mathbf{r}-\mathbf{r}')\; \bar{\chi}^{\beta}(\mathbf{r}')\Delta^{-j}\chi^{\beta}(\mathbf{r}')$$
$$= \frac{T}{2}\int \bar{\chi}^{\alpha}(\mathbf{r})\Delta^{j}\chi^{\beta}(\mathbf{r}')\; V_0(\mathbf{r}-\mathbf{r}')\; \bar{\chi}^{\beta}(\mathbf{r}')\Delta^{-j}\chi^{\alpha}(\mathbf{r})$$

- After replacement χ->ψ expressions are no longer equal!

$$S'^{(1)}_{int} = -\frac{T}{2}\int d\mathbf{r}d\mathbf{r}'\; \bar{\psi}^{\alpha}(\mathbf{r})\Delta^{j}\psi^{\alpha}(\mathbf{r})\; V_0(\mathbf{r}-\mathbf{r}')\; \bar{\psi}^{\beta}(\mathbf{r}')\Delta^{-j}\psi^{\beta}(\mathbf{r}')$$
$$S'^{(2)}_{int} = \frac{T}{2}\int d\mathbf{r}d\mathbf{r}'\; \bar{\psi}^{\alpha}(\mathbf{r})\Delta^{j}\psi^{\beta}(\mathbf{r}')\; V_0(\mathbf{r}-\mathbf{r}')\; \bar{\psi}^{\beta}(\mathbf{r}')\Delta^{-j}\psi^{\alpha}(\mathbf{r})$$

For leading order contribution in the interaction both should be used and for each of them regions of small momentum transfer should be singled out.

Model

- We propose the following model

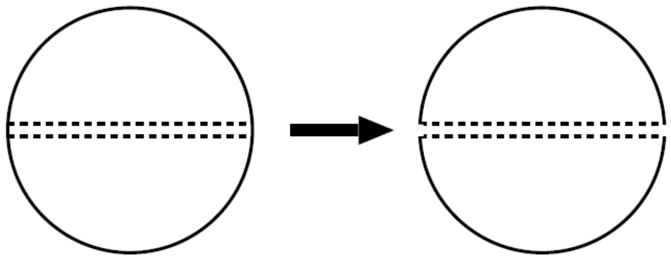
$$Z' = \int D(\psi, \bar{\psi}) \, \exp(-S'_0 - S'_{int}), \qquad S'_{int} = S'^{(1)}_{int} + S'^{(2)}_{int}$$

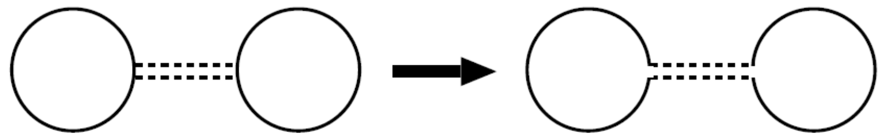
$$S'_0[\psi, \bar{\psi}, U] = -i \int d\mathbf{r} \; \bar{\psi}_\alpha(\mathbf{r}) \left[iE + \hat{\xi}_p + U(\mathbf{r}) \right] \psi_\alpha(\mathbf{r})$$

$$S'^{(1)}_{int} = -\frac{T}{2} \int d\mathbf{r} d\mathbf{r}' \; \bar{\psi}^\alpha(\mathbf{r}) \Delta^j \psi^\alpha(\mathbf{r}) \; V_0(\mathbf{r} - \mathbf{r}') \; \bar{\psi}^\beta(\mathbf{r}') \Delta^{-j} \psi^\beta(\mathbf{r}')$$

$$S'^{(2)}_{int} = \frac{T}{2} \int d\mathbf{r} d\mathbf{r}' \; \bar{\psi}^\alpha(\mathbf{r}) \Delta^j \psi^\beta(\mathbf{r}') \; V_0(\mathbf{r} - \mathbf{r}') \; \bar{\psi}^\beta(\mathbf{r}') \Delta^{-j} \psi^\alpha(\mathbf{r})$$

- Diagrammatic representation
  - Green's function of the non-interacting theory
  - Interaction-line corresponding to $S'^{(1)}_{int}$
  - Interaction-line corresponding to $S'^{(2)}_{int}$





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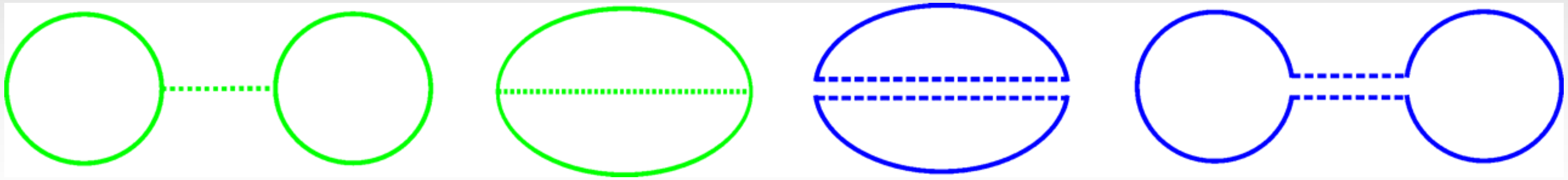
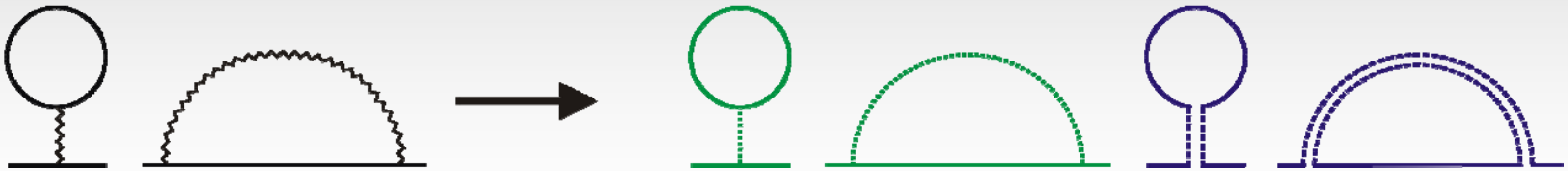
- Normalization: Vacuum loops



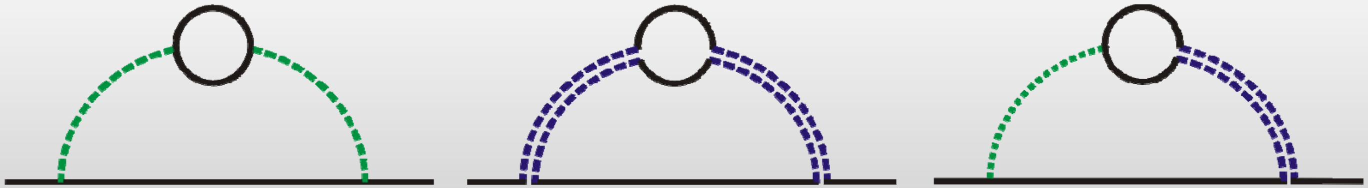
- More generally: All vacuum loops vanish due to supersymmetry.
- Disorder independent normalization is ensured.
- Disorder averaging does not pose a problem.

- Interaction corrections to Green's functions



First and fourth diagram vanish, second and third give Fock and Hartree contributions.

- Screening of the interaction lines inside these diagrams not correctly reproduced.



First and second diagram vanish, third diagram gives twice the desired contribution.

Sigma model with interaction

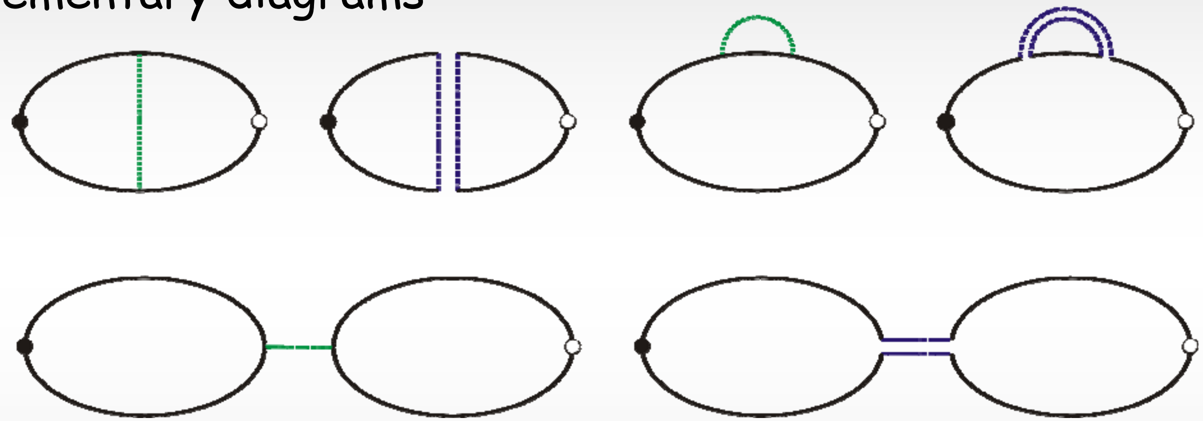
- After disorder averaging and decoupling:

$$Z' = \int D(\psi,\bar\psi)DQ\ \exp\left(-\frac{\pi\nu}{4\tau}\int d\mathbf{r}\ \mathrm{Str}\, Q^2(\mathbf{r})\right)$$
$$\times\exp\left(i\int d\mathbf{r}\ \bar\psi(\mathbf{r})\left[iE+\hat\xi_p+\frac{i}{2\tau}Q(\mathbf{r})\right]\psi(\mathbf{r})\right)\ \exp\left(-S'_{int}[\psi,\bar\psi]\right).$$

- Single out slow modes for each interaction term and consider simplest approximation exp(- < S'_{int}>)
- Amplitudes Γ_1 and Γ_2 describe processes with small and large angle scattering.
- Use saddle point g(r,r)=$\pi\nu$ Q(r).
- The sigma-model

$$F=\frac{\pi\nu}{4}\int d\mathbf{r}\ \left(\mathrm{Str}\left[D(\nabla Q)^2-4EQ\right]+\left[\Gamma_2 Q\gamma_2 Q-\Gamma_1 Q\gamma_1 Q\right]\right)$$
$$Q\gamma_1 Q = 2\pi T\sum_{n_i\alpha\beta}\left(\mathrm{Str}\left[Q^{\alpha\alpha}_{n_1n_2}\right]\mathrm{Str}\left[Q^{\beta\beta}_{n_3n_4}\right]+\mathrm{Str}\left[Q^{\alpha\alpha}_{n_1n_2}Q^{\beta\beta}_{n_3n_4}\right]\right)\delta_{n_1+n_3,n_2+n_4}$$
$$Q\gamma_2 Q = 2\pi T\sum_{n_i\alpha\beta}\left(\mathrm{Str}\left[Q^{\alpha\beta}_{n_1n_2}\right]\mathrm{Str}\left[Q^{\beta\alpha}_{n_3n_4}\right]+\mathrm{Str}\left[Q^{\alpha\beta}_{n_1n_2}Q^{\beta\alpha}_{n_3n_4}\right]\right)\delta_{n_1+n_3,n_2+n_4}$$

- Source fields and density-density correlation
  - Correlation functions can be calculated by introducing source fields.
  - A few elementary diagrams



The source terms: $F_{\vartheta} = i\pi\nu \int d\mathbf{r}\, \mathrm{Str}\, [\underline{\vartheta} Q] \leftrightarrow \circ \qquad F_{\varphi} = i\pi\nu \int d\mathbf{r}\, \mathrm{Str}\, [k_{+} \underline{\varphi} Q] \leftrightarrow \bullet,$

k_+ projects onto fermion fields and $\underline{\varphi}(\mathbf{r}) = T \sum_j \varphi(\mathbf{r}, j) \Delta^j$

- Calculation of the density-density correlation function in the ladder approximation

Testing the model

- RG (following Finkel'stein's scheme)
  - One finds from the supersymmetric model

$$\frac{1}{t}\frac{dt}{d\ln\lambda^{-1}} = \frac{t}{z}(\Gamma_1 - 2\Gamma_2), \qquad \frac{1}{t}\frac{dz}{d\ln\lambda^{-1}} = -(\Gamma_1 - 2\Gamma_2)$$

$$\frac{1}{t}\frac{d\Gamma_1}{d\ln\lambda^{-1}} = \Gamma_2, \qquad \frac{1}{t}\frac{d\Gamma_2}{d\ln\lambda^{-1}} = \Gamma_1$$

  - Equations valid
    - at first order in $t=((2\pi)^2\nu\, D)^{-1}$ to leading order Γ_i for $\Gamma_i \ll z$.
    - In this approximation they coincide with the known RG equations for short range interactions.

- Altshuler-Aronov corrections to conductivity and DOS at first order in the interaction can also be obtained by perturbation theory from the model.

Conclusion and discussion

- We propose a sigma-model for disordered systems with interaction (G.S., K. Efetov, PRB 2005).
- It is an artificially crafted model that is applicable for weak short range interactions.
- 4x4 supermatrices carry two spin and two Matsubara indices -> Structure is (considerably) more complicated than in the standard supersymmetric approach.
- We checked the model by perturbation theory and RG calculation.
- Possible applications to 0D and 1D systems, where supersymmetry has been applied for non-interacting systems. Example: Localization in disordered interacting wires.